The Theory of Alpha Decay

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October 24, 2023

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Physical Description

the process

- alpha decay starts with the nucleus of ${}^{234}_{92}U$(Uranium).
- the formula form

$$ {}^{A+4}_{Z+2}X \rightarrow {}^{4}_{2}He^{++} + {}^{A}_{Z}Y^{--} $$

- X: parent nucleus, Y: daughter nucleus exactly

$$ {}^{234}_{92}U \rightarrow {}^{4}_{2}He + {}^{230}_{90}Th(Thorium) $$

basic approximation

- the alpha particle: a well-defined separate entity and trapped in the confines of the parent nucleus (${}^{234}_{92}U$)
  $\rightarrow$ a simplified 2-body problem
- the potential $V(r)$ (different models) $\rightarrow$ the Schrodinger equation $\rightarrow$ the dynamics of the alpha particle(the daughter nucleus)

Models

- The potential $V(r)$: considered to be produced collectively by the nucleons of the daughter nucleus

General nature: three regions

- $V(r)$ produces an attractive force for α within $0 \leq r \leq R$
- Well away from the nucleus, $V(r) = 2Ze'^2/r$ for $r \gg 1fm$
- In the intermediate region, the 2 types of interaction are comparable

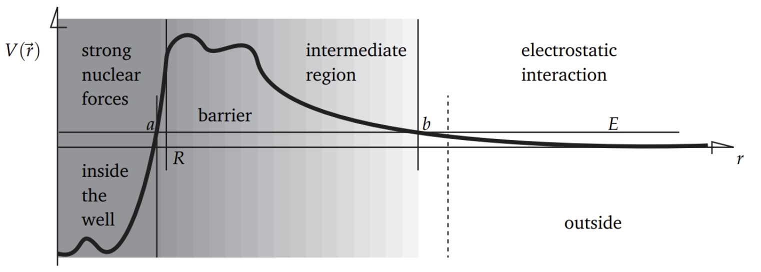

Figure 1: a sketch of the shape of $V(r)$

Models

simple model: spinless decay

Gamow's simple model(2 assumptions)

- $V(r) = -V_0$, for $0 \leq r \leq R$ & $V(r) = 2Ze'^2/r$, for $r > R \rightarrow f_{NN}$ vanishes outside (the radius of nucleus)

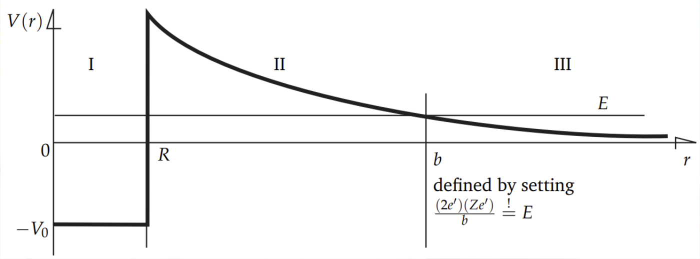

Figure 2: Gamow's simple model

Defect:

- The potential changes discontinuously at $r = R \rightarrow$ infinite force

Models

simple model: spinless decay

By letting $l = 0$, we solve the Schrodinger equation

Solutions

- Region I: $u_I(r) = A sin(kr + \delta), \quad k = \sqrt{\frac{2m_\alpha}{\hbar^2}(E + V_0)}$
- Region II: $u_{II}(r) = \frac{C}{\sqrt{\kappa(r)}} e^{-\int_R^r dr \kappa(r)} + \frac{D}{\sqrt{\kappa(r)}} e^{\int_R^r dr \kappa(r)}, \quad R < r < b$
- Region III:$u_{III}(r) \frac{A'}{\sqrt{k(r)}} e^{i \int_b^r dr k(r)} (\text{decay}) + \frac{B'}{\sqrt{k(r)}} e^{-i \int_b^r dr k(r)} (\text{capture}), \quad b < r$

where

$$\kappa(r) = \sqrt{\frac{2m_\alpha}{\hbar^2}(\frac{2Ze'^2}{r} - E)}, \quad k(r) = \sqrt{\frac{2m_\alpha}{\hbar^2}(E - \frac{2Ze'^2}{r})}$$

i.e. the factor

$$\sqrt{\frac{2m_\alpha}{\hbar^2}|E - V|}$$

Models

simple model: spinless decay

Solutions(coefficients)

By imposing the standard boundary conditions:

$$\lim_{r\to R_-} u_I(r) = \lim_{r\to R_+} u_{II}(r), \quad \lim_{r\to R_-} u_I'(r) = \lim_{r\to R_+} u_{II}'(r)$$

we get all the coefficients (where $\theta = e^{i\pi/4}$)

$$C = \frac{A}{2\sqrt{\kappa_R}}[\kappa_R sin(kR) - kcos(kR)] \tag{1}$$

$$D = \frac{A}{2\sqrt{\kappa_R}}[\kappa_R sin(kR) + kcos(kR)] \tag{2}$$

$$A' = \frac{\theta e^{\sigma} Acos(kR)}{4\sqrt{\kappa_R}}[\kappa_R tan(kR) - k - 2ie^{2\sigma}[\kappa_R tan(kR) + k]] \tag{3}$$

$$B' = \frac{\theta^* e^{\sigma} Acos(kR)}{4\sqrt{\kappa_R}}[\kappa_R tan(kR) - k + 2ie^{2\sigma}[\kappa_R tan(kR) + k]] \tag{4}$$

Models

simple model: spinless decay

Solutions(true bound states)

- We set the coefficient $D = 0$ in $u_{II}(r)$

$$D = \frac{A}{2\sqrt{\kappa_R}}[\kappa_R sin(kR) + kcos(kR)] = 0$$

- And get

$$tan(kR) = -k/\kappa_R$$

- That is the transcendental equation determining the discrete values of bound energies E.

Exact solution

general cases: $l \neq 0$

Based on Gamow's simplified model, the radial equation within the range of $0 < r < R$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[k^2 - \frac{l(l+1)}{r^2}\right]R = 0$$

$\rightarrow$ bessel equation, we have

$$R_{in}(r) = Aj_l(kr)$$

For $r > R$, by derivation, we have

$$R_{out}(r) = r^l e^{-\beta r}\left[B_1F_1(l+1+w, 2l+2; 2\beta r) + Cr^{-2l-1}{}_1F_1(w-l, -2l; 2\beta r)\right]$$

where $\beta = -2m_\alpha E/\hbar, w = \dfrac{2Ze'^2m_\alpha}{\beta\hbar^2}$

Confluent hypergeometric functions

Asymptotic form: $R_{out}(r) \sim Fr^{-(w+1)}e^{-\beta r} + Gr^{w-1}e^{+\beta r}$

Models

elaborated model: a smoothing intermediate region

In order to get a smoothing intermediate region, we introduce harmonic oscillator

elaborated potential

$$V(r) = \begin{cases} \frac{1}{2} m_\alpha \omega^2 r^2 & 0 \le r \le R \\ W - \frac{1}{2} m_\alpha v^2 (r - r_0)^2 & R \le r \le R' \\ \frac{2Ze'^2}{r} & r \ge R' \end{cases}$$

where $V(r), V'(r)$ are continuous at $r = R, R'$ by choosing proper value of W, v, r_0, R'

Models

elaborated model: a smoothing intermediate region

Sketch

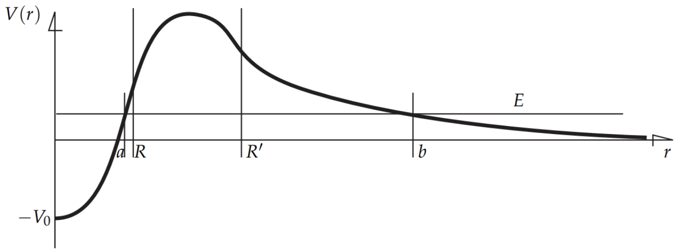

Figure 3: The shape of $V(r)$ has been refined.The inside potential has been assumed to be of the (spherical) harmonic kind.

By using the solutions to the 3-d harmonic oscillator, the analytic continuation, and the analyzed confluent hypergeometric functions in the 3 regions, the WKB methods can be used to estimate the decay rate and the stationary states.

Models

elaborated model: a smoothing intermediate region

The decay rate would involve -2σ and

$$\sigma = \sqrt{\frac{2m_\alpha}{\hbar^2}} \int_a^b dr \sqrt{V(r) - E}$$

Do not depend on the internal details!

Conclusion

1. WKB approximation is typically accurate for the decay rate (estimated by the Gamow factor)
2. The model should capture 2 characteristics of the process:
    - the coulomb replusion between the daughter nucleus and the alpha particle
    - the negligible dependence on the details of the potential inside the nucleus

Thank You!