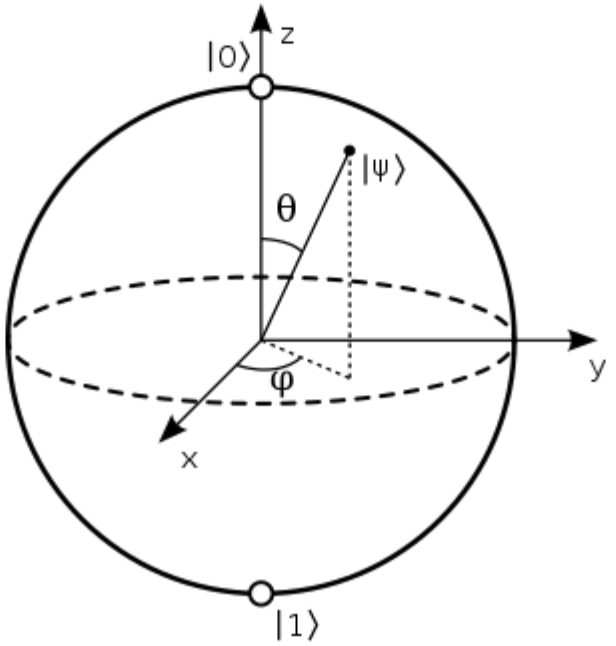
量子计算
Quantum Computing

武亦文 5.12

量子比特的表示——Bloch Sphere Representation



$$|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$|a|^2 + |b|^2 = 1$$

Single qubit states that are not entangled and lack global phase can be represented as **points on the surface of the Bloch sphere**, written as

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

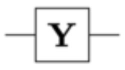
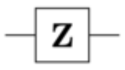
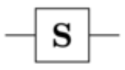
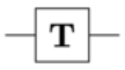
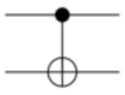
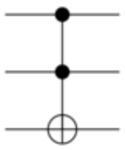
量子逻辑门
Quantum logic gate

1.Quantum logic gates are represented by **unitary matrices**.

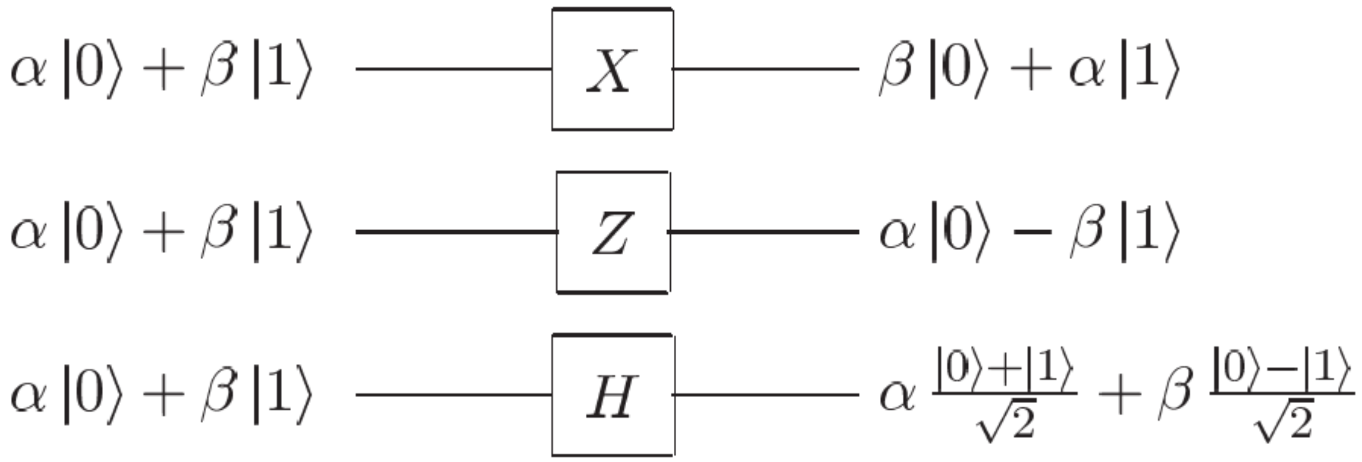
$$U\binom{a}{b} = \binom{c}{d}$$

2. unitary quantum gates are always **invertible.**

$$\binom{a}{b} = U^{-1}\binom{c}{d}$$

Operator	Gate(s)	Matrix
Pauli-X (X)		$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$
Pauli-Y (Y)		$\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$
Pauli-Z (Z)		$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$
Hadamard (H)		$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$
Phase (S, P)		$\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix}$
$\pi/8$ (T)		$\begin{bmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{bmatrix}$
Controlled Not (CNOT, CX)		$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$
Controlled Z (CZ)		$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}$
SWAP		$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$
Toffoli (CCNOT, CCX, TOFF)		$\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\end{bmatrix}$

单量子比特门
Single qubit gates



Pauli-X (X) | **Pauli-Y (Y)** | **Pauli-Z (Z)**

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

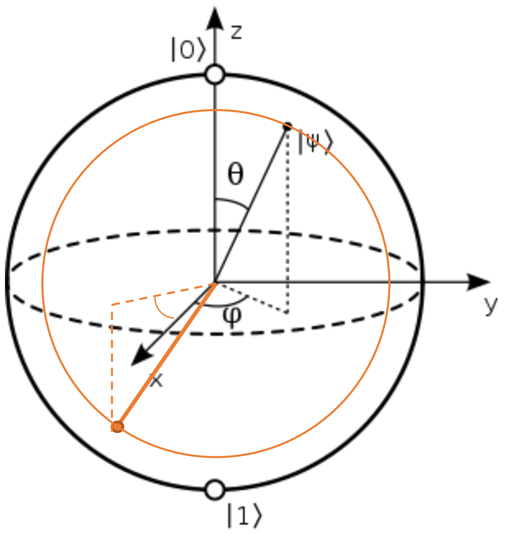
$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

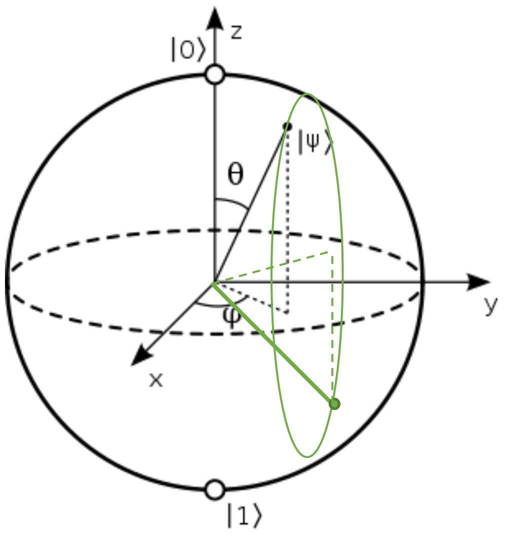
$$X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

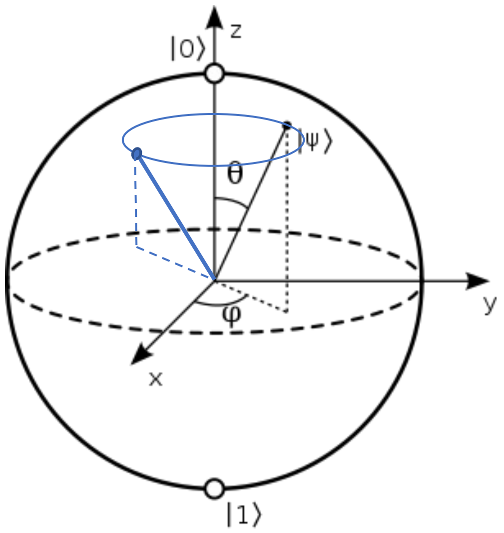
$$Y|\psi\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -ib \\ ia \end{pmatrix}$$

$$Z|\psi\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -b \end{pmatrix}$$

A rotation about $\hat{x}$ or $\hat{y}$ or $\hat{z}$ axis by 180°







Hadamard (H)

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{a+b}{\sqrt{2}} \\ \frac{a-b}{\sqrt{2}} \end{pmatrix}$$

$|0\rangle \leftrightarrow |+\rangle$
$|1\rangle \leftrightarrow |-\rangle$

A rotation about $\hat{y}$ axis by 90°, followed by a rotation about $\hat{x}$ by 180°

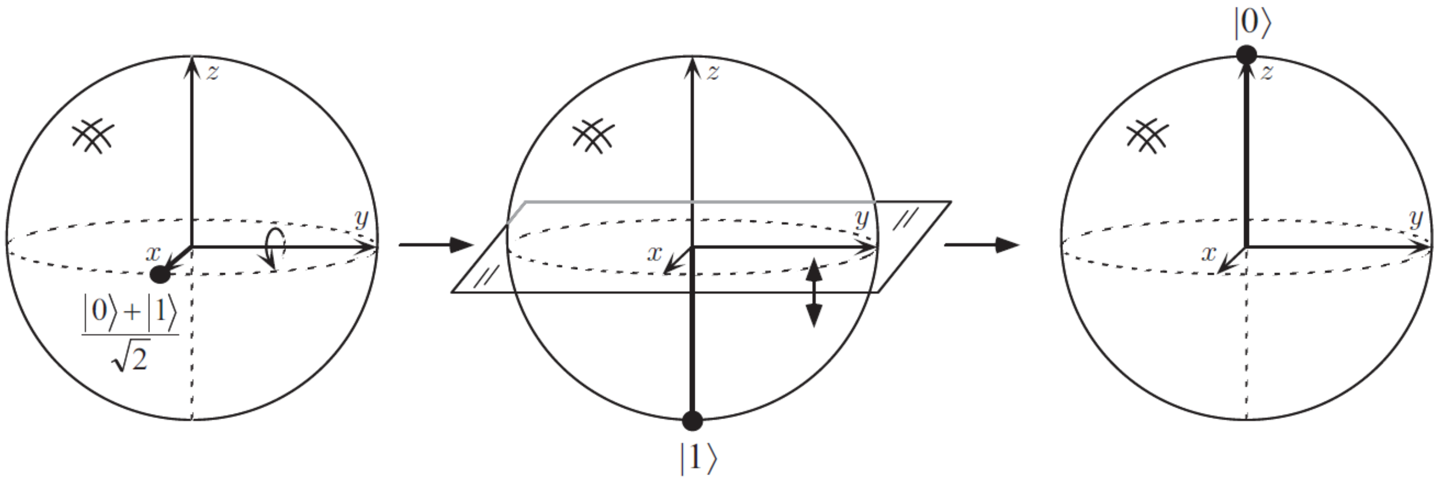


Figure 1.4. Visualization of the Hadamard gate on the Bloch sphere, acting on the input state $(|0\rangle + |1\rangle)/\sqrt{2}$.

Rotation operators

Rotation about $\hat{x}$ *or* $\hat{y}$ or $\hat{z}$ axis by θ degree

$$R_x(\theta) = e^{-\frac{i\theta X}{2}} = \cos\left(\frac{\theta}{2}\right) I - isin\left(\frac{\theta}{2}\right) X = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -isin\left(\frac{\theta}{2}\right) \\ -isin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$R_y(\theta) = e^{-\frac{i\theta Y}{2}} = \cos\left(\frac{\theta}{2}\right) I - isin\left(\frac{\theta}{2}\right) Y = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -sin\left(\frac{\theta}{2}\right) \\ sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

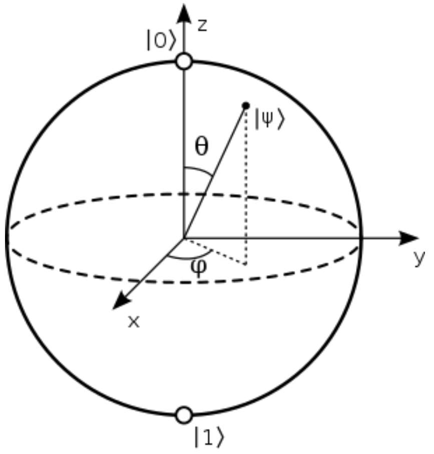
$$R_z(\theta) = e^{-\frac{i\theta Z}{2}} = \cos\left(\frac{\theta}{2}\right) I - isin\left(\frac{\theta}{2}\right) Z = \begin{pmatrix} e^{-\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{pmatrix}$$

$$\exp(ixA) = \cos(x)\, I + isin(x) A$$
$$(A^2 = I)$$

Proof:

$$\begin{aligned} &\exp(ixA) \\ &= 1 + ixA + \frac{1}{2!}(ixA)^2 + \frac{1}{3!}(ixA)^3 + \cdots \\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots \\ &+ iA\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\ &= \cos(x)\, I + isin(x) A \end{aligned}$$

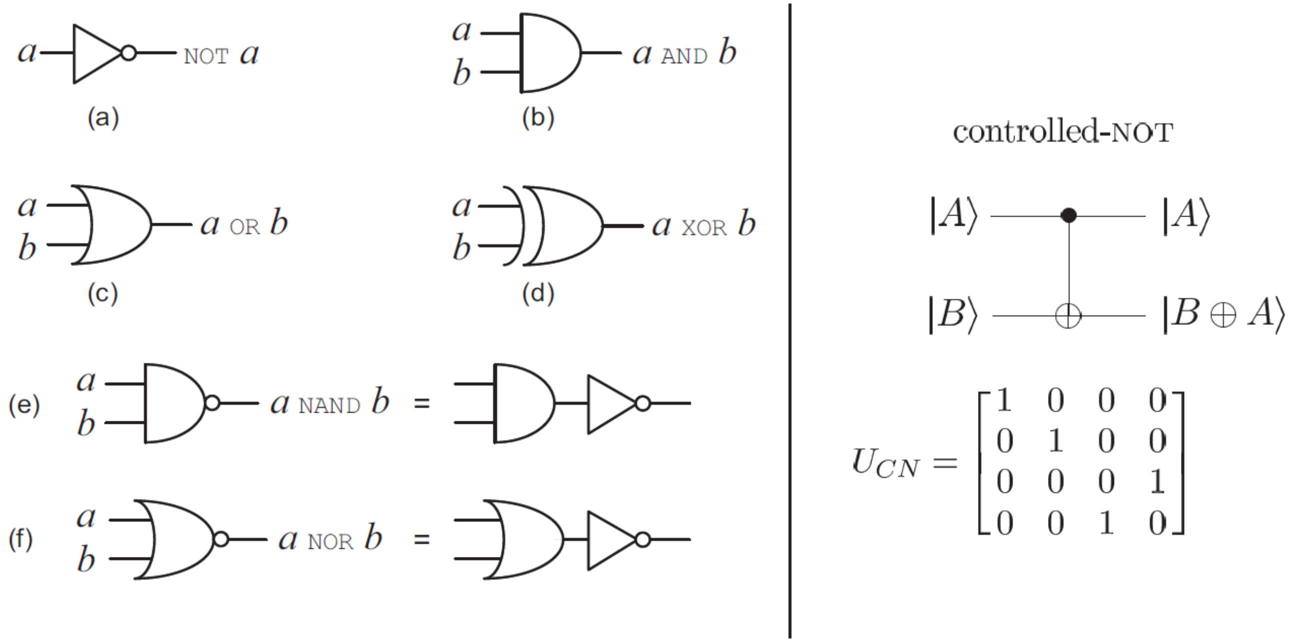
单量子比特门的分解
Decomposing single qubit operations



An arbitrary 2×2 unitary matrix may be decomposed as

$$U = e^{i\alpha} R_z(\theta) R_y(\gamma) R_z(\delta) = e^{i\alpha} \begin{pmatrix} e^{-\frac{i\beta}{2}} & 0 \\ 0 & e^{\frac{i\beta}{2}} \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\gamma}{2}\right) & -\sin\left(\frac{\gamma}{2}\right) \\ \sin\left(\frac{\gamma}{2}\right) & \cos\left(\frac{\gamma}{2}\right) \end{pmatrix} \begin{pmatrix} e^{-\frac{i\delta}{2}} & 0 \\ 0 & e^{\frac{i\delta}{2}} \end{pmatrix}$$

多量子比特门
Multiple qubit gates



$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Figure 1.6. On the left are some standard single and multiple bit gates, while on the right is the prototypical multiple qubit gate, the controlled-NOT. The matrix representation of the controlled-NOT, U_{CN}, is written with respect to the amplitudes for $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$, in that order.

$$|00\rangle \rightarrow |00\rangle$$
$$|01\rangle \rightarrow |01\rangle$$
$$|10\rangle \rightarrow |11\rangle$$
$$|11\rangle \rightarrow |10\rangle$$

Any multiple qubit logic gate may be composed from CNOT and single qubit gates.

Controlled-U gate

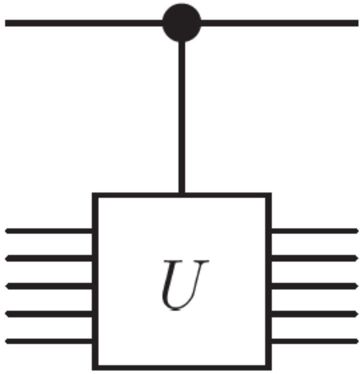


Figure 1.8. Controlled-U gate.

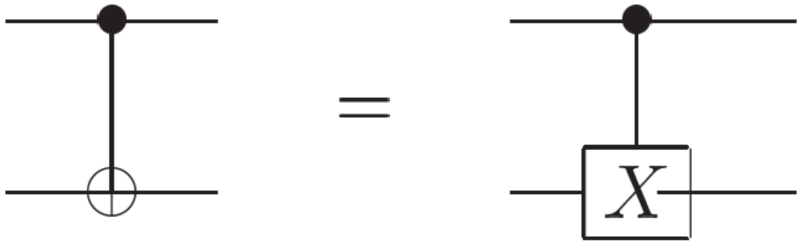


Figure 1.9. Two different representations for the controlled-NOT.

Measurement operation

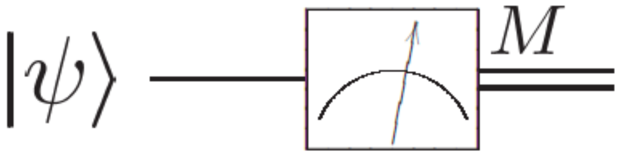


Figure 1.10. Quantum circuit symbol for measurement.

量子态可以复制吗？
Qubit copying circuit?

$$|\psi\rangle|0\rangle = a|00\rangle + b|10\rangle$$

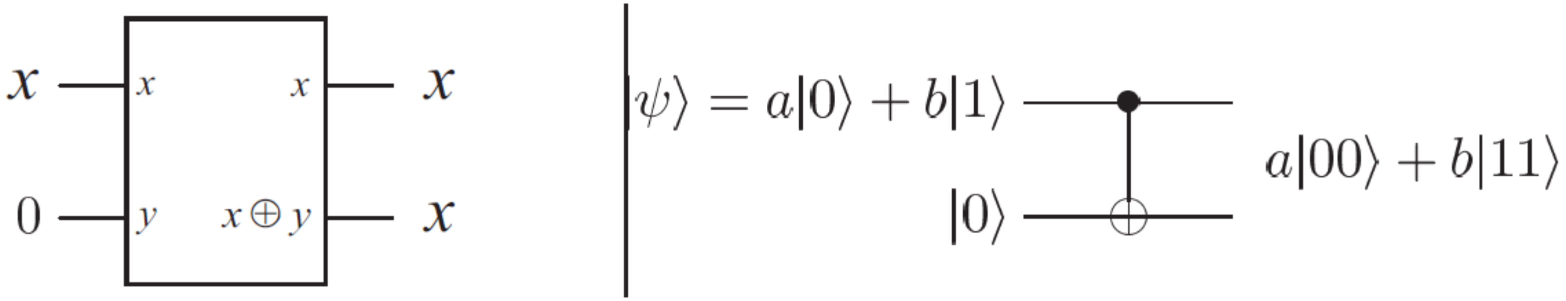


Figure 1.11. Classical and quantum circuits to 'copy' an unknown bit or qubit.

$$|\psi\rangle|\psi\rangle = a^2|00\rangle + ab|01\rangle + ab|10\rangle + b^2|11\rangle$$
$$= a|0\rangle(a|0\rangle + b|1\rangle) + b|1\rangle(a|0\rangle + b|1\rangle)$$

It is impossible to make a copy of an unknown quantum state.

Bell states (EPR pairs)

In	Out
$\|00\rangle$	$(\|00\rangle + \|11\rangle)/\sqrt{2} \equiv \|\beta_{00}\rangle$
$\|01\rangle$	$(\|01\rangle + \|10\rangle)/\sqrt{2} \equiv \|\beta_{01}\rangle$
$\|10\rangle$	$(\|00\rangle - \|11\rangle)/\sqrt{2} \equiv \|\beta_{10}\rangle$
$\|11\rangle$	$(\|01\rangle - \|10\rangle)/\sqrt{2} \equiv \|\beta_{11}\rangle$

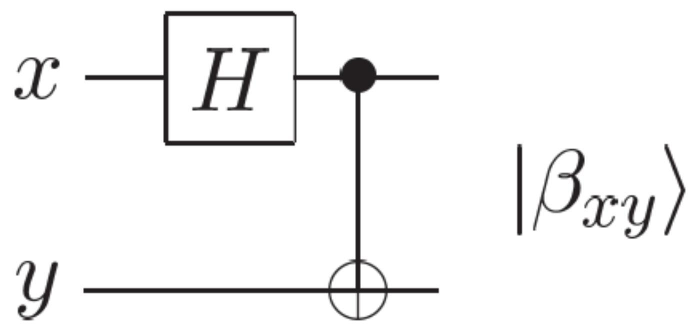


Figure 1.12. Quantum circuit to create Bell states, and its input–ouput quantum 'truth table'.

quantum teleportation

*Quantum teleportation utilizes the entangled **EPR pair** in order to send **an unknown qubit $|\psi\rangle$** to Bob, with only a small overhead of **classical communication**.*

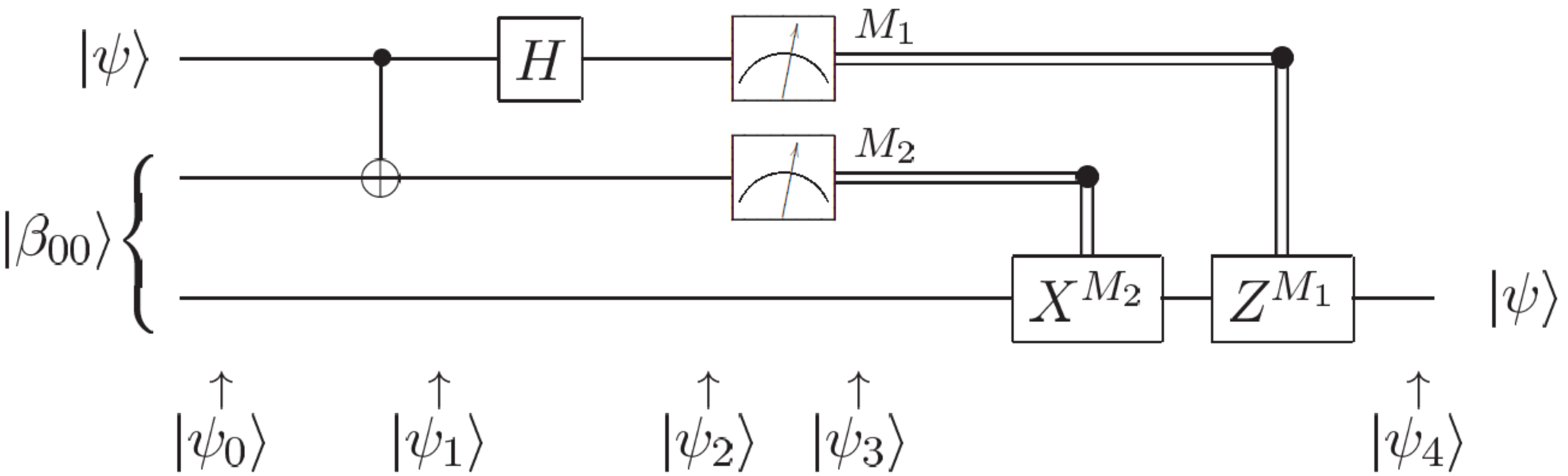


Figure 1.13. Quantum circuit for teleporting a qubit. The two top lines represent Alice's system, while the bottom line is Bob's system. The meters represent measurement, and the double lines coming out of them carry classical bits (recall that single lines denote qubits).

$$|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle = \frac{1}{\sqrt{2}}[a|0\rangle(|00\rangle + |11\rangle) + b|1\rangle(|00\rangle + |11\rangle)]$$

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}[a|0\rangle(|00\rangle + |11\rangle) + b|1\rangle(|10\rangle + |01\rangle)]$$

$$|\psi_2\rangle = \frac{1}{2}[a(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + b(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)]$$
$$= \frac{1}{2}[|00\rangle(a|0\rangle + b|1\rangle) + |01\rangle(a|1\rangle + b|0\rangle)$$
$$+|10\rangle(a|0\rangle - b|1\rangle) + |11\rangle(a|1\rangle - b|0\rangle)]$$

$$00 \rightarrow |\psi_3\rangle = (a|0\rangle + b|1\rangle)$$
$$01 \rightarrow |\psi_3\rangle = (a|1\rangle + b|0\rangle)$$
$$10 \rightarrow |\psi_3\rangle = (a|0\rangle - b|1\rangle)$$
$$11 \rightarrow |\psi_3\rangle = (a|1\rangle - b|0\rangle)$$

$$|\psi_4\rangle = (a|0\rangle + b|1\rangle) = |\psi\rangle$$