Jump to a general state of Alice and Bob. (A, B) - $\boxed{\text{Open System}}$

任一个二元状态、

$|\psi\rangle_{AB} = \sum_{i,\mu} a_{i\mu}\, |i\rangle_A \otimes |\mu\rangle_B$ ，归一化：$\sum_{i,\mu} |a_{i\mu}|^2 = 1$ (Root)

Alice 只能了解 A 空间的信息，对A中态做操作（不知道是A态还是总的 $|\psi\rangle_{AB}$）

Alice 用测量算符 M_A / $M_A \otimes I_B$

$${}_{AB}\langle\psi| M_A \otimes I_B |\psi\rangle_{AB} = \left[\sum_{j,\nu} a_{j\nu}^* \left({}_A\langle j| \otimes {}_B\langle \nu|\right)\right](M_A \otimes I_B)\left[\sum_{i,\mu} a_{i\mu}\left(|i\rangle_A \otimes |\mu\rangle_B\right)\right]$$

~~X 由 $\{|i\rangle \otimes |\mu\rangle\}$ 的正交性化简~~

记得张量积内积的运算法则

$$= \sum_{j,\nu}\sum_{i,\mu} a_{j\nu}^* a_{i\mu}\, {}_A\langle j|M_A|i\rangle_A\, {}_B\langle \nu|\mu\rangle_B$$

使用正交性.

$$= \sum_{i,j} a_{j\mu}^* a_{i\mu}\, {}_A\langle j|M_A|i\rangle_A$$

引入一个算符 Density Operator. ρ.

对A态、

$$\rho_A = \sum_{i,j}\left(\sum_\mu a_{j\mu}^* a_{i\mu}\right)|i\rangle\langle j| \qquad (\rho 0).$$

对比 (Root) 式 即 $|\psi\rangle_{AB}$ 表达式，可以发现.

~~$\rho_A = \psi$~~

$$|\psi\rangle_{AB}\,{}_{AB}\langle\psi| = \sum_{i,\mu}\sum_{j,\nu} a_{j\nu}^* a_{i\mu}\left(|i\rangle_A\,{}_A\langle j|\right)\otimes\left(|\mu\rangle_B\,{}_A\langle\nu|\right)$$

即发现.

$$\rho_A = \sum_\mu \langle\mu|\psi\rangle\langle\psi|\mu\rangle = tr_B(|\psi\rangle\langle\psi|).$$

有什么用呢？接着看，代入 ($\rho 0$) 式 即 ρ_A 的展开形式，

$$\rho_A M_A = \sum_{k,j}\left(\sum_\mu a_{j\mu}^* a_{k\mu}\right)|k\rangle\langle j| M_A.$$

发现 "!"

$$tr_A(\rho_A M_A) = \sum_i \langle i|\rho_A M_A|i\rangle = \sum_{i,j}\sum_\mu a_{j\mu}^* a_{i\mu}\langle j|M_A|i\rangle = {}_{AB}\langle\psi|M_A\otimes I_B|\psi\rangle_{AB}$$

总结.

对于一个 $|\psi\rangle = |\psi\rangle_{AB}$.

定义 $\rho_A := tr_B(|\psi\rangle\langle\psi|)$

则有 $\boxed{\langle\psi|M_A \otimes I_B|\psi\rangle = tr_A(\rho_A M_A)}$

显然 A B 是完全平权的.

$\rho_B = tr_A(|\psi\rangle\langle\psi|)$.

$\langle\psi| I_A \otimes M_B|\psi\rangle = tr_B(\rho_B M_B)$

密度矩阵（density operator）的性质.

1. Hermitian. $\rho=\rho^\dagger$

2. Nonnegative $\langle\phi|\rho|\phi\rangle\geq 0$

$$\parallel \quad \sum_{i,j,\mu} a_{j\mu}^* a_{i\mu}\langle\phi|i\rangle\langle j|\phi\rangle=\sum_\mu\left|\sum_i a_{i\mu}\langle\phi|i\rangle\right|^2\geq 0$$

3. Unit trace. $\mathrm{tr}\,\rho=\sum_{i,\mu}a_{i\mu}^* a_{j\mu}=\sum_{i,\mu}|a_{i\mu}|^2=1.$

ρ在一组正交归一基 ~~$|a\rangle$~~ $\{|a\rangle\}$ 下.可对角化.

$$\rho=\sum_a p_a|a\rangle\langle a|,\quad p_a\geq 0,\ \sum_a p_a=1.$$

考虑 $M_A=|a'\rangle\langle a'|$ $\quad\langle\psi|M_A\otimes I_B|\psi\rangle=\mathrm{tr}_A(\rho M_A)=\mathrm{tr}\left(\sum_a p_a|a\rangle\delta_{aa'}\langle a'|\right)$

即. p_a的物理含义是 Alice 制备 $|a\rangle$态的概率. $\quad=\sum_a p_{a'}\langle a|a'\rangle\langle a'|a\rangle$

$\quad=p_{a'}.$

如果 仅有一个 ≥ 0 的本征值 p_a 即 $\#\{p_a\}=1$. 则说A是 pure.; 否则 mixed.

~~下面证明. $|\psi\rangle_{AB}=$~~

下面寻找 B中一组特殊的基.

一组 $\{|i\rangle\}_A$. 正交归一. $\quad\rho_A=\sum_i p_i|i\rangle\langle i|$. (*). （总是有的）.

对于一个 $|\psi\rangle_{AB}$, B任一组正交归一基$\{|\mu\rangle_B\}$

$$|\psi\rangle_{AB}=\sum a_{i\mu}|i\rangle_A\otimes|\mu\rangle_B\quad(|\psi\rangle\text{的普遍写法}).$$

引入. $|\tilde{i}\rangle=\sum_\mu a_{i\mu}|\mu\rangle_B$ （不一定有任何性质）. 或

有 $\mathrm{tr}_B(|\tilde{i}\rangle\langle\tilde{j}|)=\sum_\mu a_{j\mu}^* a_{i\mu}=\langle\tilde{j}|\tilde{i}\rangle.$

此时 $|\psi\rangle$ 写作. $|\psi\rangle_{AB}=\sum_i|i\rangle_A\otimes|\tilde{i}\rangle_B$

ρ_A 写作 $\quad\rho_A=\mathrm{tr}_B(|\psi\rangle\langle\psi|)$ ~~$=\sum_{i,j}|i\rangle_A\langle$~~

$$=\mathrm{tr}_B\left[\sum_{ij}(|i\rangle\langle j|)_A(|\tilde{i}\rangle\langle\tilde{j}|)_B\right]\ \cancel{=\sum_{ij}(|i\rangle\langle j|)_A\langle\tilde{j}|\tilde{i}\rangle}.$$

$$=\sum_\mu\sum_{ij}(|i\rangle\langle j|)_A(\langle\mu|\tilde{i}\rangle\langle\tilde{j}|\mu\rangle)_B=\sum_{ij}(|i\rangle\langle j|)_A\langle\tilde{j}|\tilde{i}\rangle$$

对比 (*). $\langle\tilde{j}|\tilde{i}\rangle=\delta_{ij}p_i\rightarrow|i'\rangle_B=\frac{1}{\sqrt{p_i}}|\tilde{i}\rangle_B.$

$|i'\rangle_B$ 即为我们寻找的基.(正交归一).

$|\psi\rangle_{AB} = \sum_i \sqrt{p_i}\, |i\rangle_A \otimes |i'\rangle_B$ $\longleftarrow$ 完成了施密特分解.

$\#\{p_i\}$: Schmidt rank.

$\rho_A = tr_B(|\psi\rangle\langle\psi|) = \sum_i p_i\, |i\rangle\langle i|$

$\rho_B = tr_A(|\psi\rangle\langle\psi|) = \sum_i p_i\, |i'\rangle\langle i'|$

若 Schmidt rank $= 1$. $|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B$. 为 pure state.

若 · > 1 则 $|\psi\rangle_{AB}$ 是 entangled or. nonseparable

Schmit分解中的维数问题

总能找到 $\{|i\rangle\}_A$ 正交归一基 使得 $\rho_A=\sum_i p_i|i\rangle\langle i|$ (*)

对于一个 $|\psi\rangle_{AB}$，取B中任一组正交归一基 $\{|\mu\rangle\}_B$

于是 $|\psi\rangle_{AB}$ 被普遍的写成了 $|\psi\rangle_{AB}=\sum_{i,\mu}a_{i\mu}|i\rangle_A\otimes|\mu\rangle_B$

引入 $|\tilde{i}\rangle=\sum_\mu a_{i\mu}|\mu\rangle_B$ 不一定正交，不一定归一

$$\begin{pmatrix}|\tilde 1\rangle\\|\tilde 2\rangle\\|\tilde 3\rangle\\\vdots\end{pmatrix}_i=\begin{pmatrix}a_{11}&a_{12}&\cdots&\\a_{21}&\ddots&&\\\vdots&&a_{i\mu}&\\&&&\ddots\end{pmatrix}_{i\times\mu}\begin{pmatrix}|\mu_1\rangle\\|\mu_2\rangle\\|\mu_3\rangle\\\vdots\end{pmatrix}_\mu$$

讨论: $\{|i\rangle\}_A$ 和 $\{|\mu\rangle\}_B$ 是 A.B 中完备的，即 $\dim A=\#\{|i\rangle\}_A$ $\dim B=\#\{|\mu\rangle\}_B$

① $\dim[\mathrm{Span}(\{|i\rangle\}_A)]=\dim A\le\dim B=\dim[\mathrm{Span}(\{|\mu\rangle\}_B)]$

则对于矩阵 $a_{i\mu}$，$\mathrm{rank}(a_{i\mu})\le\dim A$

线性映射得到的 $\{|\tilde i\rangle\}_B$，$\#\{|\tilde i\rangle\}_B=\dim A$

$\dim[\mathrm{Span}(\{|\tilde i\rangle\}_B)]\le\#\{|\tilde i\rangle\,|\,|\tilde i\rangle\ne0,\ |\tilde i\rangle\in\{|\tilde i\rangle\}\}\le\dim A\le\dim B$

“#”表示表中元素数目

最终 $\#\{p_i\ne0\}\le\dim[\mathrm{Span}(\{|\tilde i\rangle\}_B)]$

② $\dim[\mathrm{Span}(\{|i\rangle\}_A)]=\dim A>\dim B=\dim[\mathrm{Span}(\{|\mu\rangle\}_B)]$

则对于矩阵 $a_{i\mu}$，$\mathrm{rank}(a_{i\mu})\le\dim B<\dim A$

线性映射得到的 $\{|\tilde i\rangle\}_B$，$\#\{|\tilde i\rangle\}_B=\dim A$

$\dim[\mathrm{Span}(\{|\tilde i\rangle\}_B)]\le\#\{|\tilde i\rangle\,|\,|\tilde i\rangle\ne0,\ |\tilde i\rangle\in\{|\tilde i\rangle\}\}\le\dim B<\dim A$

即可以得到 $\#\{|\tilde i\rangle\,|\,|\tilde i\rangle=0,\ |\tilde i\rangle\in\{|\tilde i\rangle\}\}\ge\dim A-\dim B$

即 $\{|\tilde i\rangle\}$ 中 0 的个数多于 $(\dim A-\dim B)$

这与最后 $\langle\tilde j|\tilde i\rangle=\delta_{ij}p_i$，$\#\{p_i\ne0\}\le\min\{\dim A,\dim B\}$ 相自恰

此时，$|\psi\rangle$ 写作 $|\psi\rangle=\sum_i|i\rangle_A\otimes|\tilde i\rangle_B$

进一步 $\rho_A=\mathrm{tr}_B(|\psi\rangle\langle\psi|)=\mathrm{tr}_B[\sum_{ij}(|i\rangle\langle j|)_A\ (|\tilde i\rangle\langle\tilde j|)_B]=\sum_\mu[\sum_{ij}(|i\rangle\langle j|)_A(\langle\mu|\tilde i\rangle_B\langle\tilde j|\mu\rangle)]$

$$=\sum_{ij}\Big[(|i\rangle\langle j|)_A\underbrace{\Big(\sum_\mu\langle\mu|\tilde i\rangle\langle\tilde j|\mu\rangle\Big)}_{\mathrm{tr}_B(|\tilde i\rangle\langle\tilde j|)}\Big]$$

代入

$|\tilde i\rangle=\sum_\mu a_{i\mu}|\mu\rangle_B\Rightarrow\mathrm{tr}_B(|\tilde i\rangle\langle\tilde j|)=\sum_\nu {}_B\langle\nu|\tilde i\rangle\langle\tilde j|\nu\rangle_B=\sum_\nu a^*_{j\nu}a_{i\nu}=\langle\tilde j|\tilde i\rangle$

得到 $\rho_A=\sum_{ij}\langle\tilde j|\tilde i\rangle(|i\rangle\langle j|)_A$

对比(*)式 $\rho_A=\sum_i p_i|i\rangle\langle i|$

得 $\langle\tilde j|\tilde i\rangle=\delta_{ij}p_i$

归一化 $|i'\rangle_B=\frac{1}{\sqrt{p_i}}|\tilde i\rangle_B\Rightarrow|\psi\rangle_{AB}=\sum_i\sqrt{p_i}|i\rangle_A\otimes|i'\rangle_B$

The ~~space~~ set of density operator is convex.


"球里"

$\rho(\lambda) = \lambda\rho_1 + (1-\lambda)\rho_2$. $\underline{0\leq\lambda\leq 1}$ 下文一直有.

厄米 ✓.

迹为1. ✓

正定? $\langle\psi|\rho(\lambda)|\psi\rangle = \lambda\langle\psi|\rho_1|\psi\rangle + (1-\lambda)\langle\psi|\rho_2|\psi\rangle \geq 0$. ✓.

$\boxed{\text{Convex}}$

对应到物理上.

$$\langle M\rangle \left(= \langle\psi|M|\psi\rangle\right) = \lambda\, tr(\rho_1 M) + (1-\lambda)\, tr\,\rho_2 M \cancel{= tr\,\rho(\lambda) M}.$$

$$= tr\left[\lambda\rho_1 + (1-\lambda)\rho_2\right]M = tr\,\rho(\lambda)M.$$

说明~~可以有物理~~ 物理上同样有 $\rho(\lambda) := \lambda\rho_1 + (1-\lambda)\rho_2$.

纯态的ρ 不可由其它态 线性表出.

证: 设 $\rho = |\psi\rangle\langle\psi| = \lambda\rho_1 + (1-\lambda)\rho_2$

有任意 $|\psi^\perp\rangle$, $\langle\psi^\perp|\psi\rangle = 0$.

$$0 = \langle\psi^\perp|\rho|\psi^\perp\rangle = \lambda\underbrace{\langle\psi^\perp|\rho_1|\psi^\perp\rangle}_{\geq 0} + (1-\lambda)\underbrace{\langle\psi^\perp|\rho_2|\psi^\perp\rangle}_{\geq 0}$$

$$\Rightarrow \langle\psi^\perp|\rho_1|\psi^\perp\rangle = 0 \quad \& \quad \langle\psi^\perp|\rho_2|\psi^\perp\rangle = 0.$$

$|\psi^\perp\rangle$ 任意 $\Rightarrow$. $\rho_1, \rho_2 \propto \rho = |\psi\rangle\langle\psi|$.

圈到 Bloch sphere 里去.

Pauli Operator $\sigma_1\ \sigma_2\ \sigma_3$. $\vec{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$.

① $\langle \hat{n}\cdot\vec{\sigma}\rangle$ 为态对应矢量在 $\hat{n}$ 上的投影 $\langle \hat{n},\vec{\sigma}\rangle = \hat{n}\cdot\vec{p}$. 态对应矢量.

② 由于 $\hat{n}\cdot\vec{\sigma}\,|\psi\rangle = |\psi\rangle$. 当 $|\psi\rangle$ 是纯态.
故. $\rho(\hat{n}) = |\psi\rangle\langle\psi| \Rightarrow (\hat{n}\cdot\vec{\sigma})\rho(\hat{n}) = \rho(\hat{n}) = \rho(\hat{n})(\hat{n}\cdot\vec{\sigma})$.

③ 正定 $\det\rho \geq 0$.

猜一个* $\rho(\vec{p}) = \frac{1}{2}(I+\vec{p}\cdot\vec{\sigma}) = \frac{1}{2}\begin{pmatrix} 1+p_3 & p_1-ip_3 \\ p_1+ip_2 & 1-p_3 \end{pmatrix}$

验证.

① $\operatorname{tr}\sigma_i\sigma_j = 2\delta_{ij}$, $\operatorname{tr}(\hat{n}\cdot\vec{\sigma}) = 0$.

$$\Rightarrow \operatorname{tr}[\rho(\vec{p})(\hat{n}\cdot\vec{\sigma})] = \operatorname{tr}\left[\tfrac{1}{2}(I+\vec{p}\cdot\vec{\sigma})(\hat{n}\cdot\vec{\sigma})\right]$$
$$\cancel{= \operatorname{tr}\left[\tfrac{1}{2}\textstyle\sum_{i,j}(I+p_i\sigma_i)(n_j\sigma_j)\right]}$$
$$= 0 + \tfrac{1}{2}\operatorname{tr}[(\vec{p}\cdot\vec{\sigma})(\hat{n}\cdot\vec{\sigma})]$$
$$= \tfrac{1}{2}\operatorname{tr}\textstyle\sum_{i,j} p_i\sigma_i n_j\sigma_j$$
$$= \textstyle\sum_i n_i p_i = \hat{n}\cdot\vec{p}. \quad \checkmark$$

② $(\hat{n}\cdot\vec{\sigma})^2 = I$, $\cancel{\Rightarrow \rho(\hat{n}) = \hat{n}}$ 结合 $\rho(\vec{p}) = \frac{1}{2}(I+\vec{p}\cdot\vec{\sigma})$.
$\rho(\hat{n}) = \frac{1}{2}(I+\hat{n}\cdot\vec{\sigma}) \Rightarrow \rho(\hat{n}) = (\hat{n}\cdot\vec{\sigma})\rho(\hat{n}) = \rho(\hat{n})(\hat{n}\cdot\vec{\sigma})$ ✓

③ $\det\rho \geq 0$.

$\frac{1}{4}(1-\vec{p}^2) \geq 0 \Rightarrow |\vec{p}| \leq 1$ 在球内. ✓

关于 $\rho(\vec{p})=\frac{1}{2}(I+\vec{p}\cdot\vec{\sigma})$ 为密度矩阵的一般形式

Slides 思路:

猜 对于纯态 $\rho(\hat{n})=\frac{1}{2}(I+\hat{n}\cdot\vec{\sigma})$

证明 / 验证 $\rho=|\psi(\hat{n})\rangle\langle\psi(\hat{n})|$ 与 $\rho(\hat{n})$ 性质完全一致

之后如下,

$\rho(\vec{p})=\frac{1}{2}(I+\vec{p}\cdot\vec{\sigma})$, $(|\vec{p}|<1)$ 为什么是混态. ?

def.

$$\rho=\sum_a p_a |a\rangle\langle a| \,,\quad \#\{p_a\}>1$$

$$=\sum_a p_a\, \rho(|a\rangle)$$

$$\sum_a p_a=1$$

即 $\rho=\sum_i p_i\, \rho_i$ — 纯态

纯态 $\rho(\hat{n}_i)=\frac{1}{2}(I+\vec{n}_i\cdot\vec{\sigma})$

$$\rho=\sum_i p_i\,\rho_i=\frac{1}{2}\Big[I+\sum_i p_i(\vec{n}_i\cdot\vec{\sigma})\Big]$$

$$=\frac{1}{2}\Big[I+\underbrace{\Big(\sum_i p_i\,\vec{n}_i\Big)}_{\vec{p}\ 矢量}\cdot\vec{\sigma}\Big]$$

即混态 $\rho(\vec{p})=\frac{1}{2}(I+\vec{p}\cdot\vec{\sigma})$

得到 ρ 的一般形式.

试图厘清一些问题.

① 为什么 A 中会有球内的态呢?

$|\psi\rangle = a|0\rangle + b|1\rangle$ 不能表示球内的态, 只能表示纯态.

混态是 Open System A.B 2个态相互作用以后得来的.

在 Page 2 写道. A中 $\rho = \sum_a p_a |a\rangle\langle a|$ 时说 p_a 的含义是Alice 制备 $|a\rangle$ 的概率

与 A.B 子系统相互作用得到.

$\boxed{\text{无法区分}}$.

所以在 Open System 里. ρ_A 比 A 更加能表示 A 中的状态.

② 什么是混态?

正交归一基

对角化后 $\rho = \sum_a p_a |a\rangle\langle a|$ $\quad \#\{p_a\} > 1$.

问. 一个混态 $|\psi\rangle_A$.

$\rho = |\psi\rangle\langle\psi|$. $\#\{p_a\} = 1$. ① 纯态还是混态?

是纯态了吧?