Scattering of Two Spinless Particles

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2-particle wave functions

- $\psi(x_1, x_2)$,$|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$
- $|\psi\rangle$: a vector in the 2-particle Hilbert space $\mathscr{H}$
- $|\phi\rangle$, $|\chi\rangle$: a vector in 1-particle space $\mathscr{H}_1$ and $\mathscr{H}_2$

scalar product & tensor product

- if $|n\rangle$ & $|m\rangle$ are orthogonal bases of the 1-particle spaces $\mathscr{H}_1$ & $\mathscr{H}_2$, then the products

$$|n\rangle \otimes |m\rangle \, (n, m = 1, 2, 3 \cdots)$$

  $\rightarrow$ an orthogonal basis of the 2-particle space $\mathscr{H}$
- in words, $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$

scalar product & tensor product

- the scalar products satisfy :

$$(\langle\phi'| \otimes \langle\chi'|)(|\phi\rangle \otimes |\chi\rangle) = \langle\phi'|\phi\rangle \langle\chi'|\chi\rangle$$

- e.g. 1 particle with the degrees of freedom of spin s concerned

$$\mathscr{H} = \mathscr{H}_{space} \otimes \mathscr{H}_{spin}$$

where $\mathscr{H}_{space}$: $\mathscr{L}^2(\mathbb{R}^3)$, $\mathscr{H}_{spin}$: $(2s+1)$-dimensional spin space, forming $(2s+1)$-component spinor wave functions

2 spinless particles

- regard $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ as a factoring of $\mathscr{H}$
- so the factoring of the space of the 2-particle wf: $\mathscr{H} = \mathscr{H}_{cm} \otimes \mathscr{H}_{rel}$

product operators

- Def: if A and B acting on $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively, operator $A \otimes B$ acting on $\mathscr{H}$ is defined by the relation

$$(A \otimes B)(|\phi\rangle \otimes |\chi\rangle) = A|\phi\rangle \otimes B|\chi\rangle$$

which is the same as $(A \otimes 1)(1 \otimes B)$

the case of 2 particles

- $H = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} + V = H^0 + V$, $H = \frac{\bar{P}^2}{2M} + [\frac{P^2}{2m} + V(x)] = H_{cm} + H_{rel}$
- where relative quantities: P, m, one of which acts only on $\mathscr{H}_{cm}$, the other on $\mathscr{H}_{rel}$

- $U(t) = e^{-iHt} = e^{-i(H_{cm}+H_{rel})t} = e^{-iH_{cm}t} \otimes e^{-iH_{rel}t}$: the motions of CM & the rel are independent
- in the same way, the free evolution operator: $U^0(t) = e^{-iH^0t} = e^{-iH_{cm}t} \otimes e^{-iH_{rel}t}$

the 2-particle S operator

- use the def $S = \Omega_-^\dagger \Omega_+$ & $\Omega_\pm = \lim_{t \to \mp\infty} U(t)^\dagger U^0(t)$
- then $U(t)^\dagger U^0(t) = 1_{cm} \otimes (e^{iH_{rel}t} e^{-iH_{rel}^0 t})$
- the same mathematical structure with that of 1-particle system with Hamiltonian $H_{rel} = \frac{P^2}{2m} + V(x) = H_{rel}^0 + V(x)$: a single particle in a fixed potential

- 2-particle moller operators acting on the 2-particle space $\mathscr{H} = \mathscr{L}^2(\mathbb{R}^6)$ have the simple form :

$$\mathbf{\Omega}_\pm = 1_{cm} \otimes \Omega_\pm$$

  where $\Omega_\pm = \lim_{t \to \mp\infty} e^{iH_{rel}t} e^{-iH^0_{rel}t}$
- then $\boldsymbol{S} = \mathbf{\Omega}^\dagger_- \mathbf{\Omega}_+ = 1_{cm} \otimes S$, where $S = \Omega^\dagger_- \Omega_+$ acts on $\mathscr{H}_{rel}$, $|\psi_{out}\rangle = \boldsymbol{S} |\psi_{in}\rangle$

conservation of energy-momentum

- $[\boldsymbol{S}, \bar{P}] = 0$
- the matrix elements $\langle p_1', p_2' | \boldsymbol{S} | p_1, p_2 \rangle$ contains the factors $\delta(E_1' + E_2' - E_1 - E_2)\delta_3(p_1' + p_2' - p_1 - p_2)$

- $|p_1, p_2\rangle = |\bar{p}, p\rangle = |\bar{p}\rangle \otimes |p\rangle$
- so $\langle p_1', p_2'| \boldsymbol{S} |p_1, p_2\rangle = \langle \bar{p}', p'| (1_{cm} \otimes S) |\bar{p}, p\rangle = \delta_3(\bar{p}' - \bar{p}) \langle p'| S |p\rangle$
- by the familar decomposition $\langle p'| S |p\rangle = \delta_3(p' - p) - 2\pi i \delta(E_{p'} - E_p) t(p' \leftarrow p)$
- combining the 2 eq, we get

$$\begin{aligned} \langle p_1', p_2'| \boldsymbol{S} |p_1, p_2\rangle =& \delta_3(p_1' - p_1)\delta_3(p_2' - p_2) \\ & - 2\pi i \delta(\sum E_i' - \sum E_i) \\ & \times \delta_3(\sum p_i' - \sum p_i) t(p' \leftarrow p) \end{aligned}$$

- similarly, $f(p' \leftarrow p) = -(2\pi)^2 m t(p' \leftarrow p)$

Thank you for your listening!