Geometry, entanglement and causality in the scattering matrix

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Tanthco Z-seminar April 12, 2022

I will discuss work done at the University of Washington in collaboration with David B. Kaplan, Natalie Klco (Caltech), Martin J. Savage, Roland C. Farrell and Mira Varma (Yale).

Based on: arXiv:1812.03138

arXiv:2011.01278

arXiv:2108.00646

arXiv:2112.02733

arXiv:2112.03472

arXiv:2112.05800

See also I.Low and T.Mehen: arXiv:2104.10835

Outline

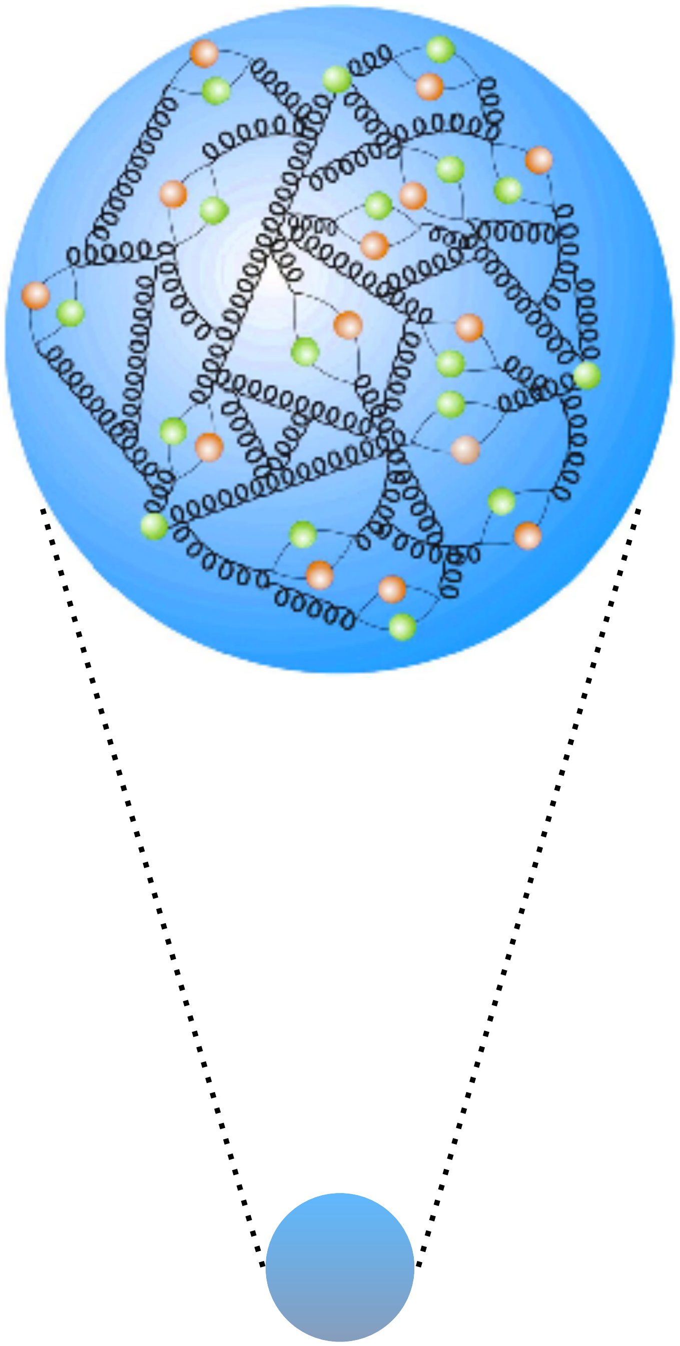
- ✓ Motivation
- ✓ Baryon-baryon scattering
- ✓ S-matrix entanglement power
- ✓ Geometric theory of the S-matrix

Motivation

Do measures of entanglement provide interesting “new” information about scattering processes?

Can we learn something about few-body interactions at low energies that is solely due to entanglement and may not be captured in effective field theory?

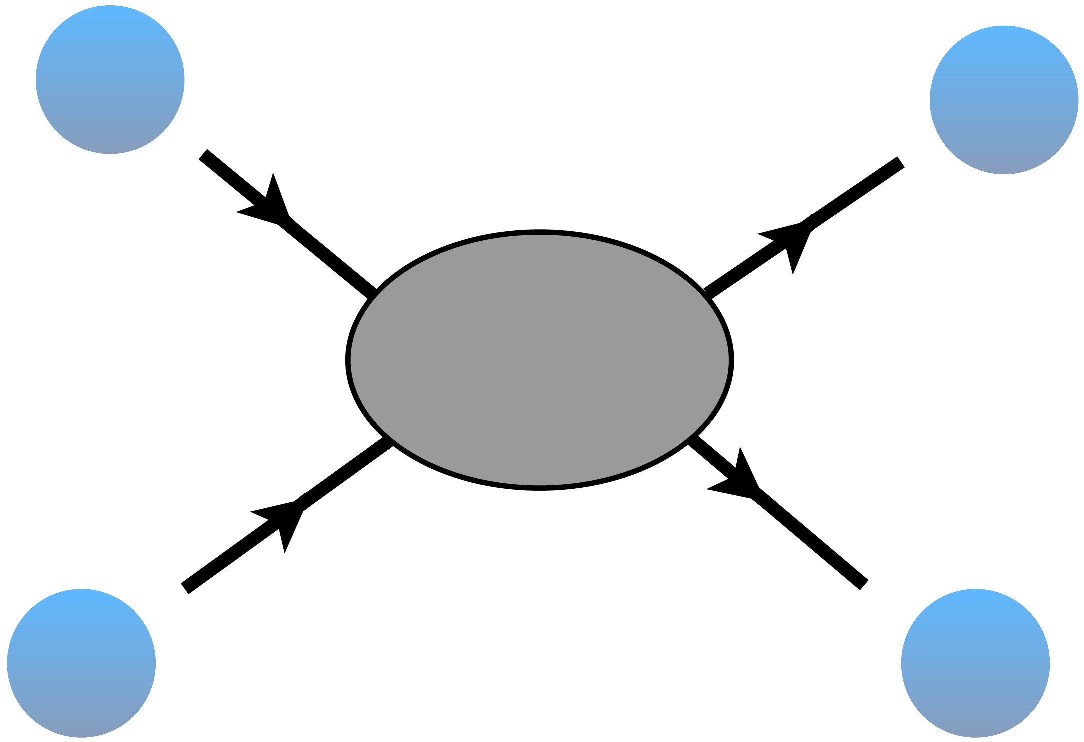
Baryon-baryon scattering



- Baryons are highly-entangled, strongly-coupled, many-body quantum systems

- Baryons at very-low energies are, to first approximation, structureless spin one-half fermions

Nucleon-nucleon scattering



At very-low energies, the unitary NN S-matrix is constrained by Fermi statistics, spin and isospin:

$$\hat{\mathbf{S}}(p) = \frac{1}{4}\left(3e^{i2\delta_1(p)} + e^{i2\delta_0(p)}\right)\hat{\mathbf{1}} + \frac{1}{4}\left(e^{i2\delta_1(p)} - e^{i2\delta_0(p)}\right)\hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{\sigma}}$$

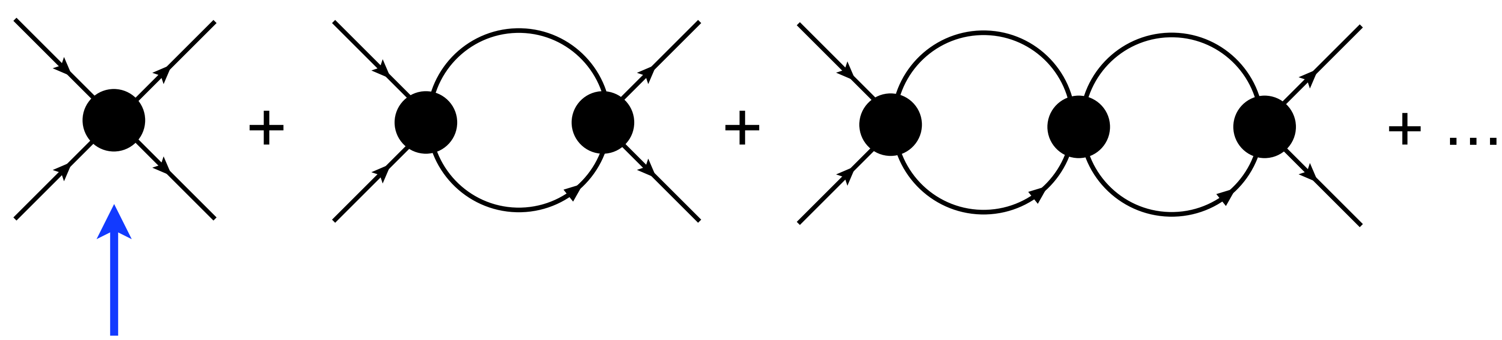
$$\delta_0 \in {}^1S_0 \qquad \delta_1 \in {}^3S_1$$

The leading order (LO) EFT `action' for NN scattering at very low energies is constrained by general principles: Galilean invariance, spin, isospin, etc.

$$\mathcal{L}_{\rm LO} = -\frac{1}{2} C_S (N^\dagger N)^2 - \frac{1}{2} C_T \left(N^\dagger \hat{\boldsymbol{\sigma}} N\right) \cdot \left(N^\dagger \hat{\boldsymbol{\sigma}} N\right)$$

NN scattering from EFT:

$$\hat{T}_S \; = $$



$C_S + C_T\, \vec{\sigma}_1 \cdot \vec{\sigma}_2$

$$\hat{S} \; = \; \hat{\mathbf{1}} \; + \; \hat{T}_S$$

Why work with the S-matrix?

- We want a unitary operator that characterizes the interaction
- The S-matrix is, in some sense, the most fundamental object for discussing entanglement

$$C_T^{\rm PDS}/C_S^{\rm PDS} = 0.0824 \qquad C_T = 0 \;\rightarrow SU(4)_W \quad 4 = \begin{pmatrix} p\uparrow \\ p\downarrow \\ n\uparrow \\ n\downarrow \end{pmatrix}$$

There is a multiplicity of explanations for this:

- Large-N QCD: $SU(4)_{SF} \rightarrow SU(4)_W$

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- Schrödinger symmetry: $SU(4)_W$

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- Vanishing spin entanglement? $SU(4)_W$

What about strange physics?

$$\mathcal{L}_{\rm LO}^{n_f=3} = -c_1\langle \underline{B_i^\dagger B_i B_j^\dagger B_j}\rangle - c_2\langle \underline{B_i^\dagger B_j B_j^\dagger B_i}\rangle$$
$$-c_3\langle \underline{B_i^\dagger B_j^\dagger B_i B_j}\rangle - c_4\langle \underline{B_i^\dagger B_j^\dagger B_j B_i}\rangle$$
$$-c_5\langle B_i^\dagger B_i\rangle\langle B_j^\dagger B_j\rangle - c_6\underline{\langle B_i^\dagger B_j\rangle\langle B_j^\dagger B_i\rangle}$$

Here lattice QCD simulations are needed!

Theory expectations:

- Large-N QCD: $SU(6)_{SF}$

$$c_1 = -\frac{7}{27}b\,,\quad c_2 = \frac{1}{9}b\,,\quad c_3 = \frac{10}{81}b$$
$$c_4 = -\frac{14}{81}b\,,\quad c_5 = a + \frac{2}{9}b\,,\quad c_6 = -\frac{1}{9}b$$

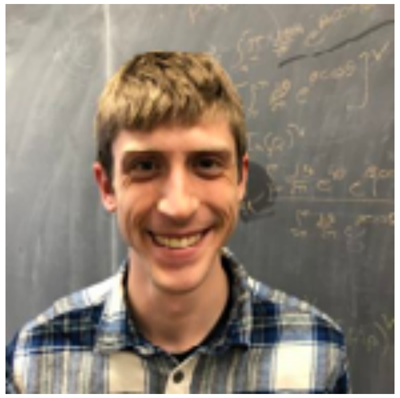
$||$

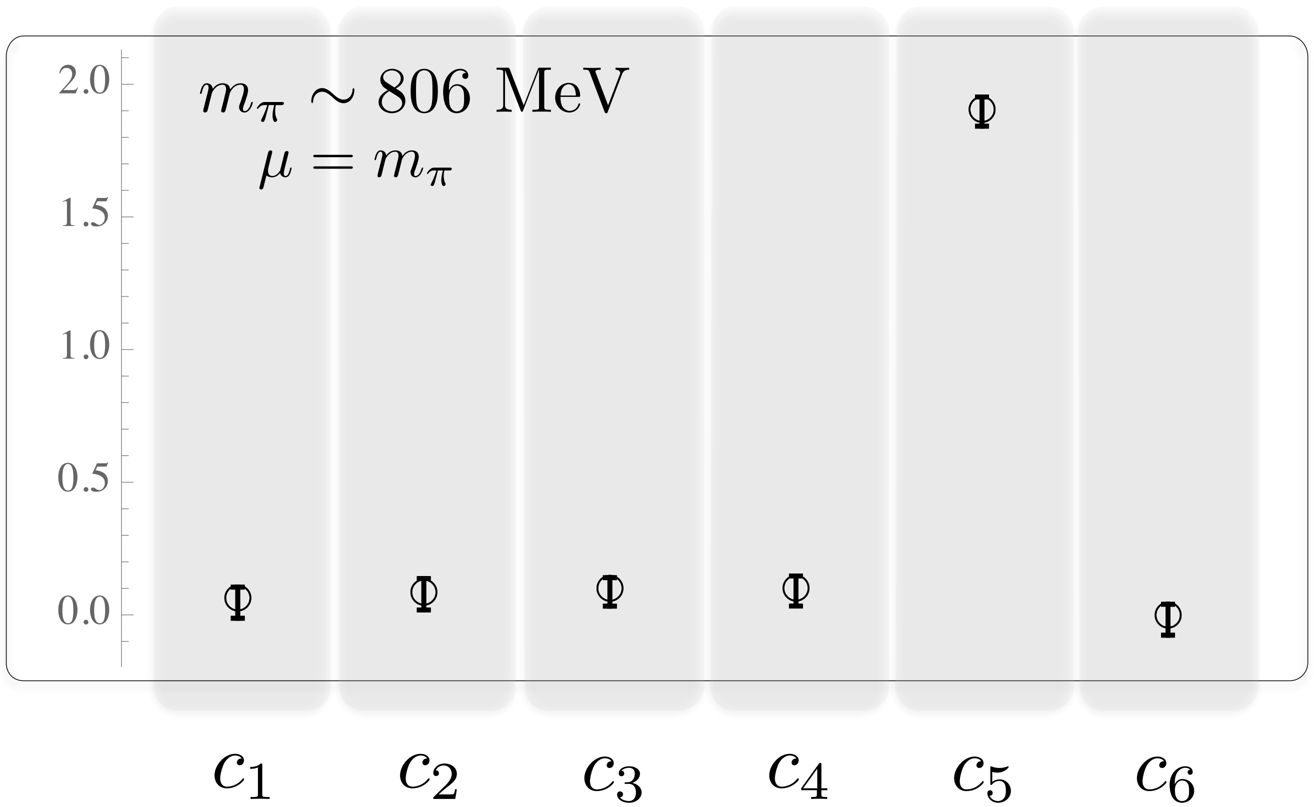
- Schrödinger symmetry: *scattering lengths near unitarity?*

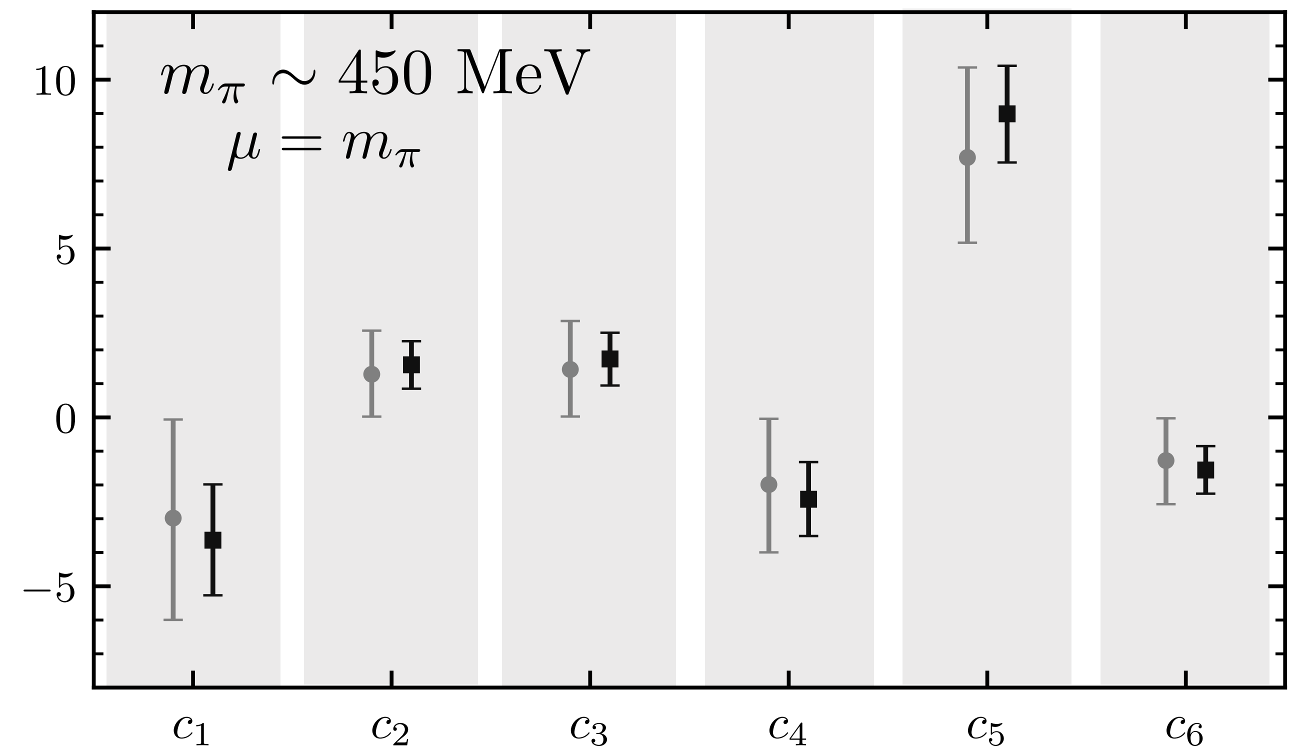
$\not||$

- Vanishing spin entanglement?

$$c_1 = c_2 = c_3 = c_4 = c_6 = 0$$
$$SU(16)_{SF}$$



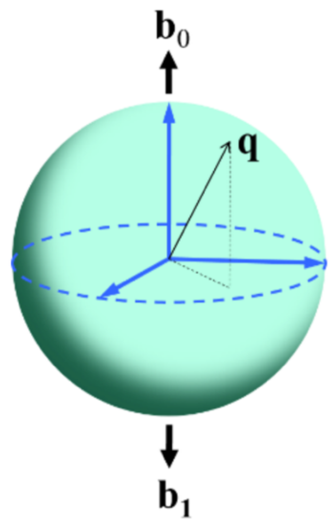




Lattice QCD agrees with predictions from vanishing entanglement:

$$SU(16)_{SF}$$

Need a measure of the entanglement of interaction

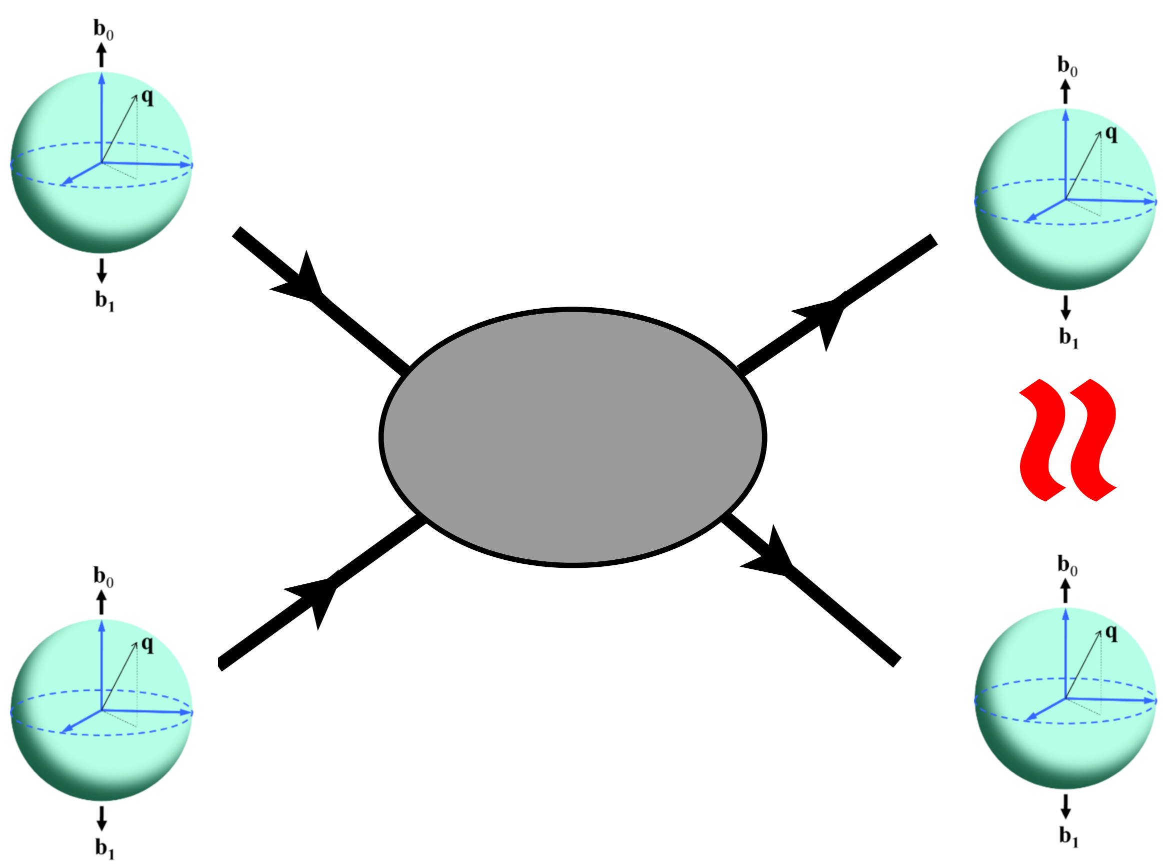


- Baryons as qubits

$$\mathbf{CP}^1 \simeq \mathbf{S}^2 \qquad \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\downarrow\rangle$$

Scattering of qubits

$$|p\uparrow\rangle \otimes |n\downarrow\rangle$$



$$\wr \quad \alpha\,|p\uparrow\rangle \otimes |n\downarrow\rangle + \beta\,|p\downarrow\rangle \otimes |n\uparrow\rangle$$

Re-express the S-matrix as:

$$\hat{\mathbf{S}} = \frac{1}{2}\left(e^{i2\delta_1} + e^{i2\delta_0}\right)\hat{\mathbf{1}} \; + \; \frac{1}{2}\left(e^{i2\delta_1} - e^{i2\delta_0}\right)\left(\hat{\mathbf{1}} + \hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{\sigma}}\right)/2$$

$$\equiv \; u\,\hat{\mathbf{1}} \; + \; v\,\mathcal{P}_{12}$$

SWAP: $\mathcal{P}_{12}\,|p\uparrow\rangle \otimes |n\downarrow\rangle \; = \; |p\downarrow\rangle \otimes |n\uparrow\rangle$

product states

$$spin\ entanglement \;\sim\; |u\,v| \;\sim\; |\sin(2(\delta_1 - \delta_0))|$$

The entanglement power (EP) of the S-matrix gives a state-independent measure of spin entanglement

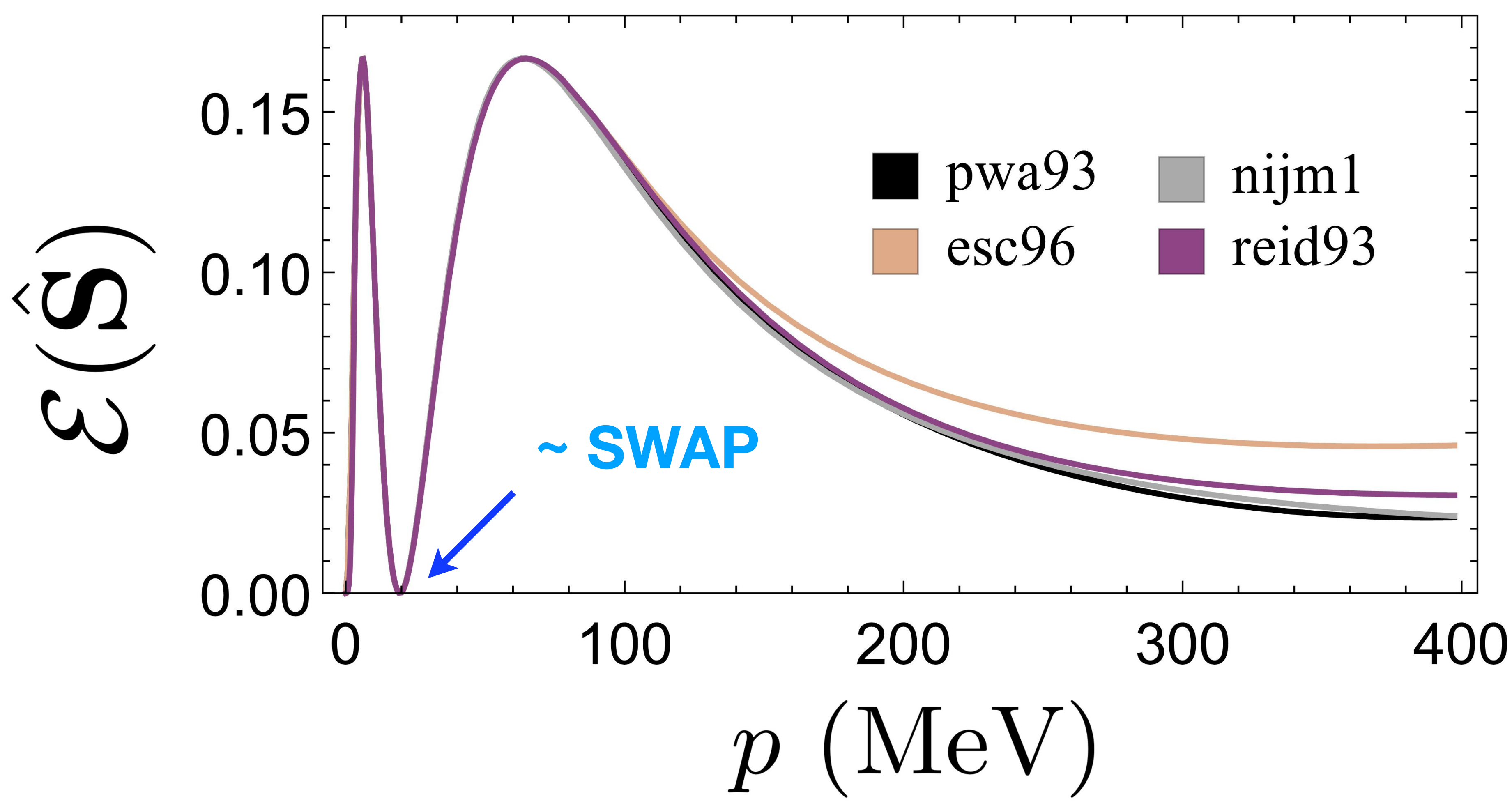
$$\mathcal{E}(\hat{\mathbf{S}}) = 1 - \int \frac{d\Omega_1}{4\pi}\,\frac{d\Omega_2}{4\pi}\;\mathrm{Tr}_1\left[\,\hat{\rho}_1^2\,\right]\mathcal{P}\left(\Omega_1,\Omega_2\right)$$

$$|\psi_{\rm out}\rangle = \hat{\mathbf{S}}|\psi_{\rm in}\rangle$$

$$\hat{\rho}_{12} = |\psi_{\rm out}\rangle\langle\psi_{\rm out}|$$

$$\hat{\rho}_1 = \mathrm{Tr}_2\left[\,\hat{\rho}_{12}\,\right]$$

$$\mathcal{E}(\hat{\mathbf{S}}) = N_{\mathcal{P}}\;\sin^2\left(2(\delta_1-\delta_0)\right)$$



Do measures of entanglement provide interesting “new” information about scattering processes?

- At threshold, minimization of the EP recovers Wigner symmetry and, in the three-flavor case, the spectral pattern observed in lattice QCD simulations, and a new SU(16) spin-flavor symmetry.

$$\mathcal{L}_{\mathrm{LO}}^{n_f=3} \to -\frac{1}{2} c_S \left(\mathcal{B}^\dagger \mathcal{B}\right)^2 \quad \mathcal{B} = \left(p_\uparrow, p_\downarrow, n_\uparrow, n_\downarrow, \Lambda_\uparrow, \ldots\right)^T$$

- Beyond threshold, minimization of the EP is observed at NLO in the EFT of NN scattering. However, the structure of the momentum dependence is mysterious…
- The EFT of NN scattering relies on the locality of the spacetime description. Is it possible to obtain the S-matrix from a spacetime-independent geometric formulation?

S-matrix basics

Consider single-channel s-wave scattering

$$S \;=\; e^{i2\delta(k)} \;=\; 1 \;-\; i\frac{kM}{2\pi}T(k)$$

Very near threshold the scattering length dominates

$$T(k) \;=\; \frac{4\pi}{M}\frac{a}{1+iak} \qquad S(k) \;=\; \frac{1-iak}{1+iak}$$

Trivial and unitary fixed points:

$$S^* = \pm 1$$

S has symmetries that are NOT symmetries of T

$$k \mapsto e^{\beta}k \qquad a \mapsto e^{-\beta}a$$

UV/IR: $k \mapsto \frac{1}{a^2 k}$

$$\delta(k) \mapsto -\delta(k) \pm \frac{\pi}{2} \quad , \quad S \to -S^*$$

Consider RG flow of EFT contact operator coupling

$$C_0(\omega) = \frac{4\pi}{M}\frac{1}{1/a - \omega}$$

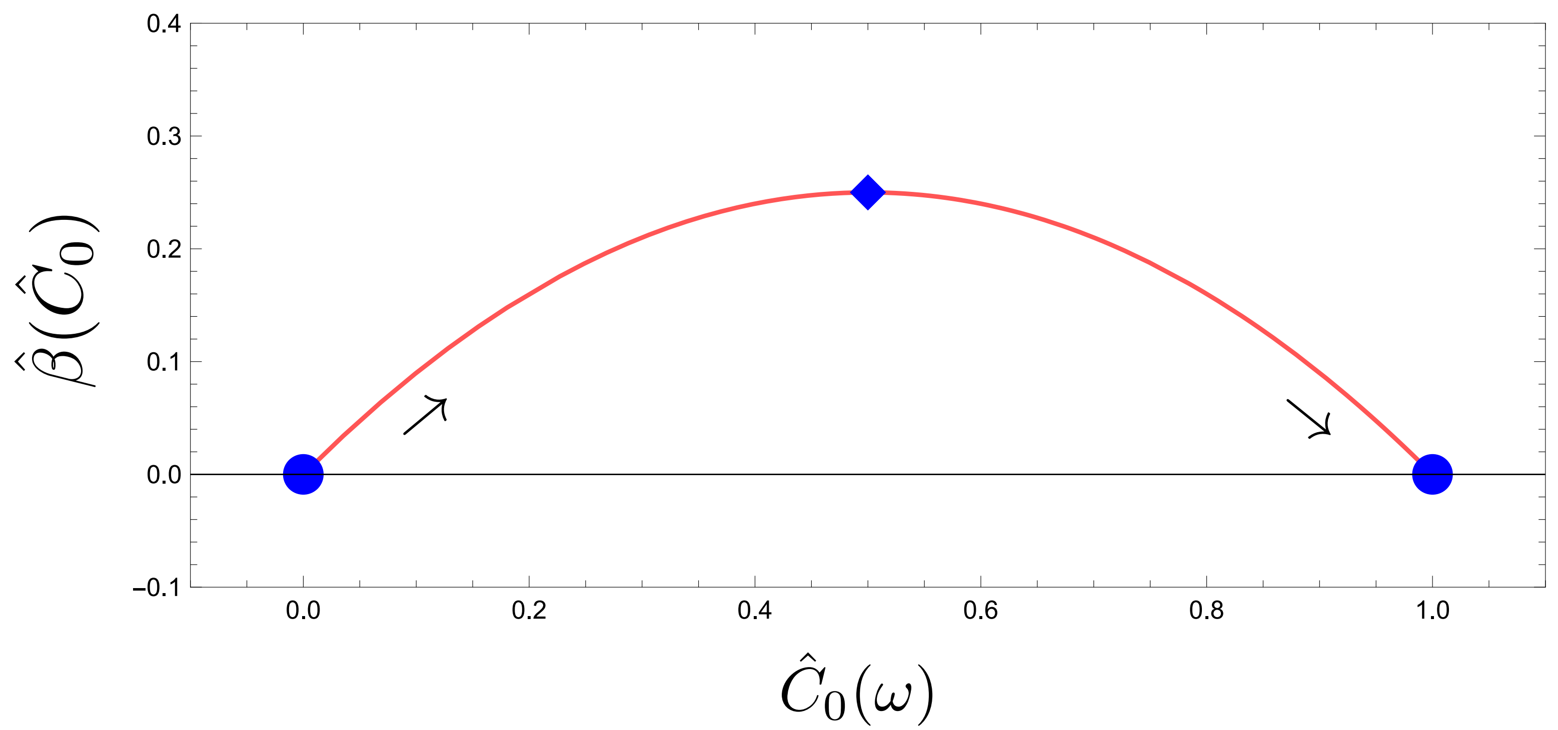
The beta function exhibits the two RG fixed points

$$\hat{\beta}(\hat{C}_0) = \omega\frac{d}{d\omega}\hat{C}_0(\omega) = -\hat{C}_0(\omega)\left(\hat{C}_0(\omega) - 1\right) \qquad \hat{C}_0 \equiv C_0/C_{0\star}$$

And is manifestly invariant w.r.t. the UV/IR symmetry

$$\omega \mapsto \frac{1}{a^2\omega}$$

$$\hat{C}_0(\omega) \leftrightarrow 1 - \hat{C}_0(\omega)$$



Geometry of the S-matrix

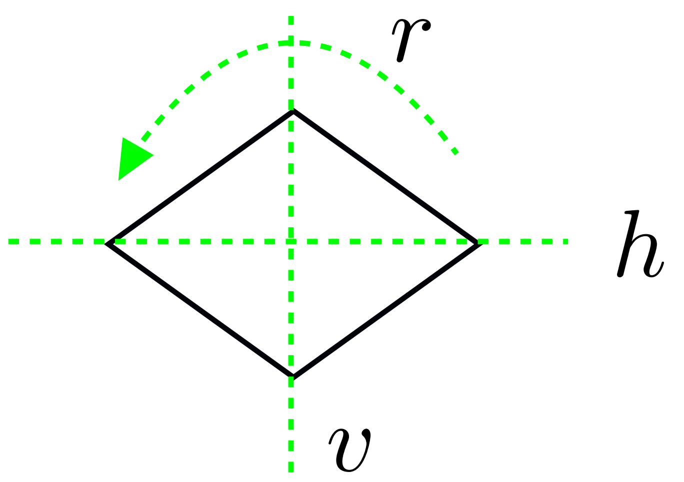
In NN s-wave scattering, there are four RG fixed points

$$\hat{\mathbf{S}}_{①} = +\hat{\mathbf{1}} \qquad \hat{\mathbf{S}}_{③} = +(\hat{\mathbf{1}} + \hat{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{\sigma}})/2$$

$$\hat{\mathbf{S}}_{②} = -\hat{\mathbf{1}} \qquad \hat{\mathbf{S}}_{④} = -(\hat{\mathbf{1}} + \hat{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{\sigma}})/2$$

They realize the Klein four-group: $\mathbb{Z}_2 \otimes \mathbb{Z}_2$

This is the symmetry group of the rhombus



At leading order (LO) in the ERE

$$u(p) = \frac{1}{2}\left(\frac{1 - ia_1 p}{1 + ia_1 p} + \frac{1 - ia_0 p}{1 + ia_0 p}\right) \quad , \quad v(p) = \frac{1}{2}\left(\frac{1 - ia_1 p}{1 + ia_1 p} - \frac{1 - ia_0 p}{1 + ia_0 p}\right)$$

Geometry of the S-matrix

S-matrix coordinates:

$$\hat{\mathbf{S}} = u(p)\,\hat{\mathbf{1}} + v(p)\,\left(\hat{\mathbf{1}} + \hat{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{\sigma}}\right)/2$$

$$u(p) = x(p) + i\,y(p) \qquad v(p) = z(p) + i\,w(p)$$

Unitarity:

$$\hat{\mathbf{S}}^{\dagger}\hat{\mathbf{S}} = \hat{\mathbf{1}}$$

$$1 = x^2 + y^2 + z^2 + w^2 \qquad S^3$$

$$0 = xz + yw$$

Parametric representation:

$$x = \tfrac{1}{2}\,r[\cos(\phi) + \cos(\theta)] \qquad y = \tfrac{1}{2}\,r[\sin(\phi) + \sin(\theta)]$$

$$z = \tfrac{1}{2}\,r[-\cos(\phi) + \cos(\theta)] \qquad w = \tfrac{1}{2}\,r[-\sin(\phi) + \sin(\theta)]$$

$$\phi \equiv 2\delta_0 \qquad \theta \equiv 2\delta_1$$

$$\phi \in [0, 2\pi] \qquad \theta \in [0, 2\pi] \qquad r = 1$$

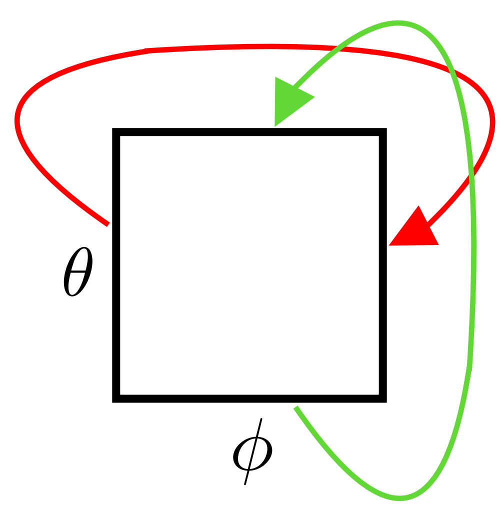
Isotropic S-matrix coordinates can be embedded in four-dimensional Euclidean space $\mathbb{R}^4$

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2$$

Compact Riemannian manifold: flat torus

$$ds^2 = \frac{1}{2}\left(d\phi^2 + d\theta^2\right)$$

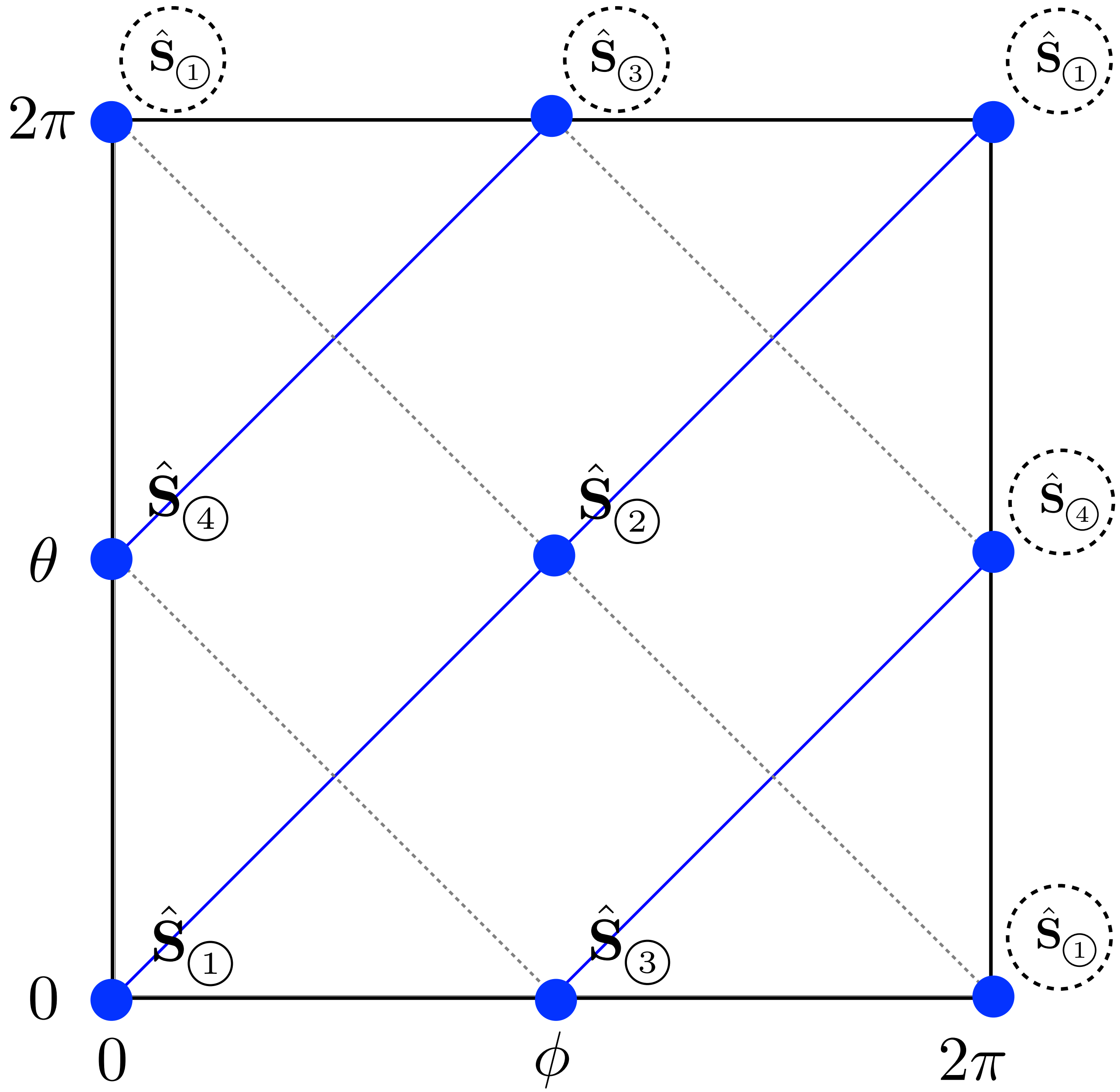
$$\mathbb{T}^2 \sim S^1 \otimes S^1$$



$$\phi \in [0, 2\pi]$$

$$\theta \in [0, 2\pi]$$

Geometry of the flat torus



Isometry group is:

$$SO(2) \otimes SO(2) \otimes \mathbb{Z}_2^6$$

$\phi \mapsto \phi + \epsilon$	$\theta \mapsto \theta + \epsilon$
$\phi \mapsto \phi + \epsilon$	$\theta \mapsto \theta - \epsilon$

$\phi \mapsto \pi - \phi$	$\theta \mapsto \pi - \theta$
$\phi \mapsto -\phi$	$\theta \mapsto -\theta$
$\phi \mapsto \theta$	$\theta \mapsto \phi$
$\phi \mapsto -\theta$	$\theta \mapsto -\phi$
$\phi \mapsto \pi - \theta$	$\theta \mapsto \pi - \phi$
$\phi \mapsto \pi + \theta$	$\theta \mapsto \pi + \phi$

With vanishing entanglement power there is no manifold:
2D surface is a lattice with fixed points as vertices and special geodesics as links

$$\mathcal{E}(\hat{\mathbf{S}}) = N_{\mathcal{P}} \sin^2(\phi - \theta)$$

Trajectories on a Riemannian manifold are governed by the action

$$\int L\left(\mathcal{X}, \dot{\mathcal{X}}\right) d\sigma \;=\; \int \left(\mathbf{N}^{-2} g_{ab} \dot{\mathcal{X}}^a \dot{\mathcal{X}}^b \;-\; \mathbb{V}(\mathcal{X})\right) \mathbf{N} d\sigma$$

Minimizing the action gives geodesics modified by a conservative force

$$\ddot{\mathcal{X}}^a \;+\; {}_g\Gamma^a{}_{bc} \dot{\mathcal{X}}^b \dot{\mathcal{X}}^c \;=\; \kappa(\sigma) \dot{\mathcal{X}}^a \;-\; \tfrac{1}{2} \mathbf{N}^2 g^{ab} \partial_b \mathbb{V}(\mathcal{X})$$

$$\kappa(\sigma) \;\equiv\; \frac{\dot{\mathbf{N}}}{\mathbf{N}} = \frac{d}{d\sigma} \ln \frac{d\lambda}{d\sigma}$$

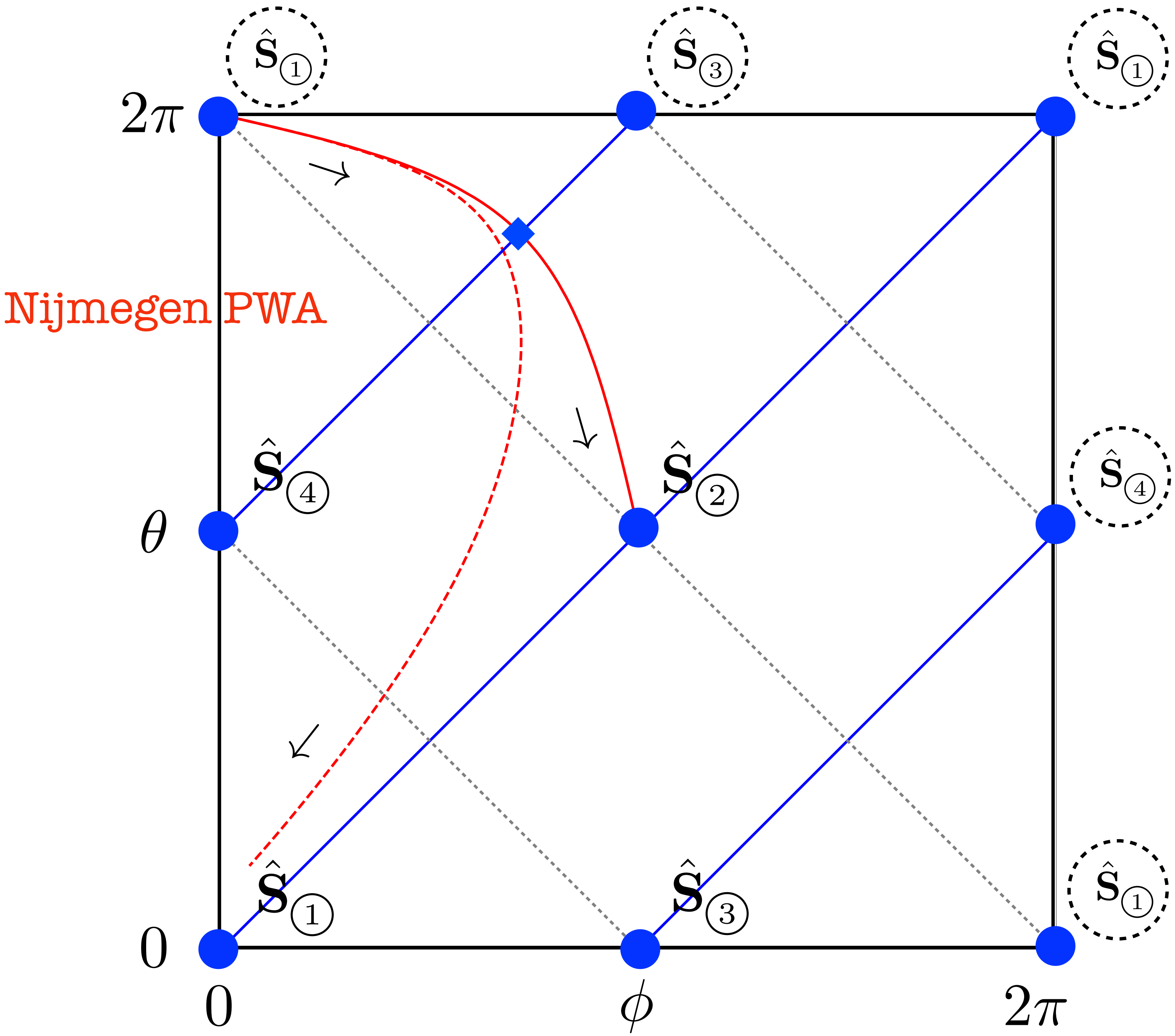
"inaffinity"

On the flat torus

$$\mathcal{X}^1 = \phi \qquad \mathcal{X}^2 = \theta \qquad \hat{g} = \tfrac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Consider leading-order in the effective-range expansion vs data

$$\phi = -2\tan^{-1}(a_0 p) \qquad \theta = -2\tan^{-1}(a_1 p)$$



Leading-order is exactly solvable with "potential"

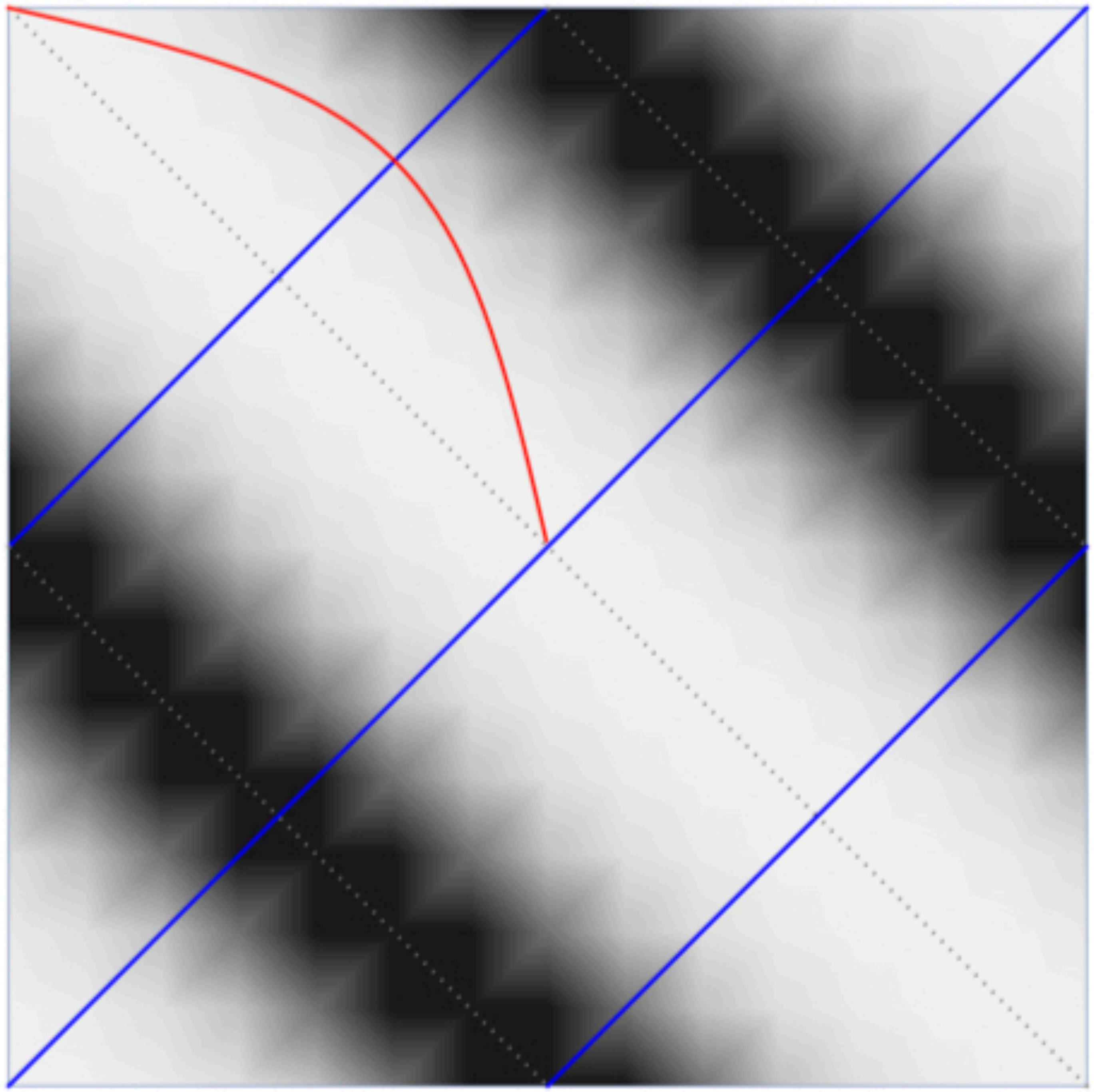
$$\mathbb{V}(\phi,\theta) = \frac{|a_0 a_1|}{(|a_0|+|a_1|)^2 c_1^2}\tan^2\left(\tfrac{1}{2}(\phi+\epsilon\,\theta)\right)$$

Note UV/IR symmetry

$$p \mapsto \frac{1}{|a_1 a_0| p}$$

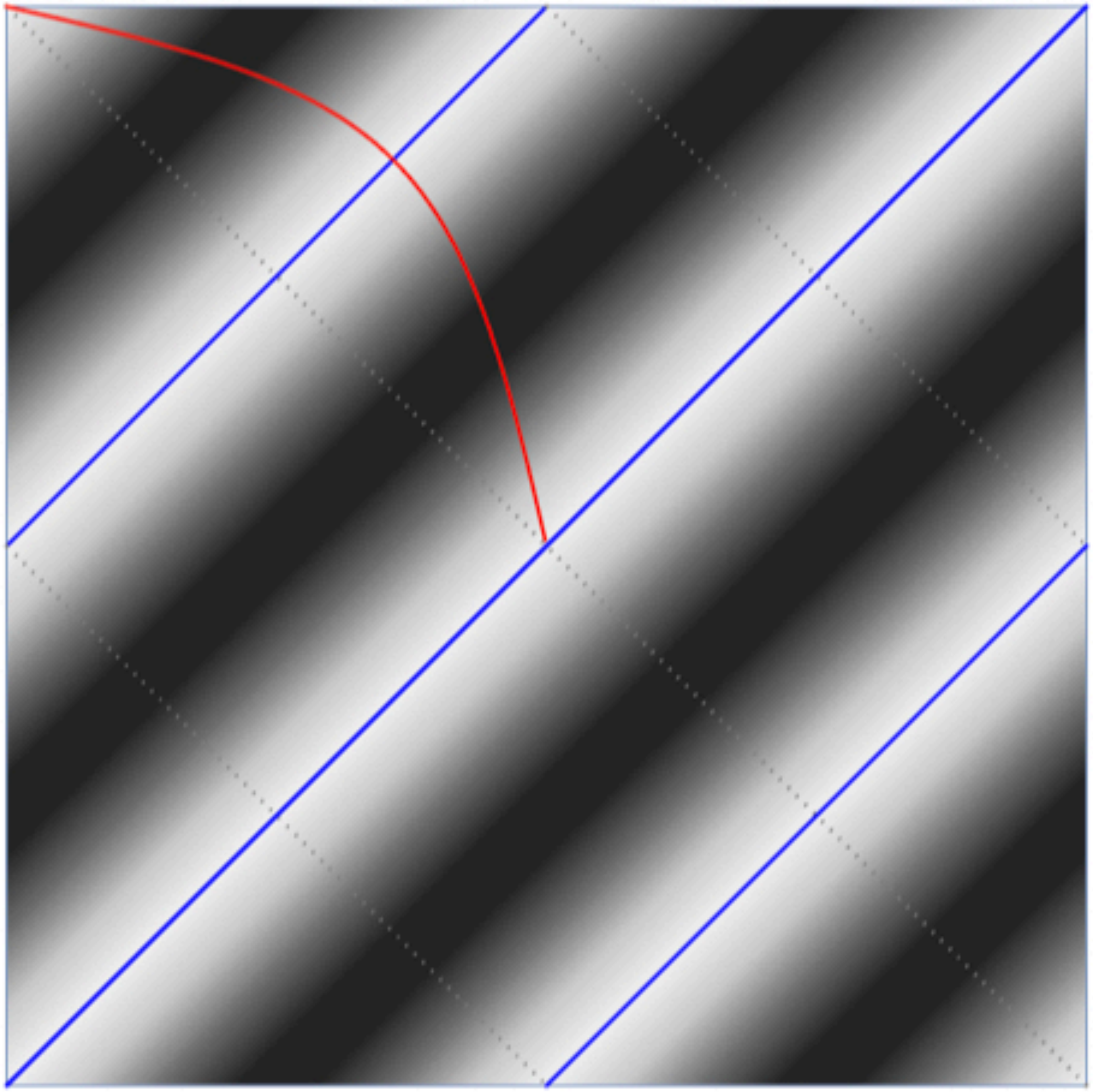
Conformal invariance!

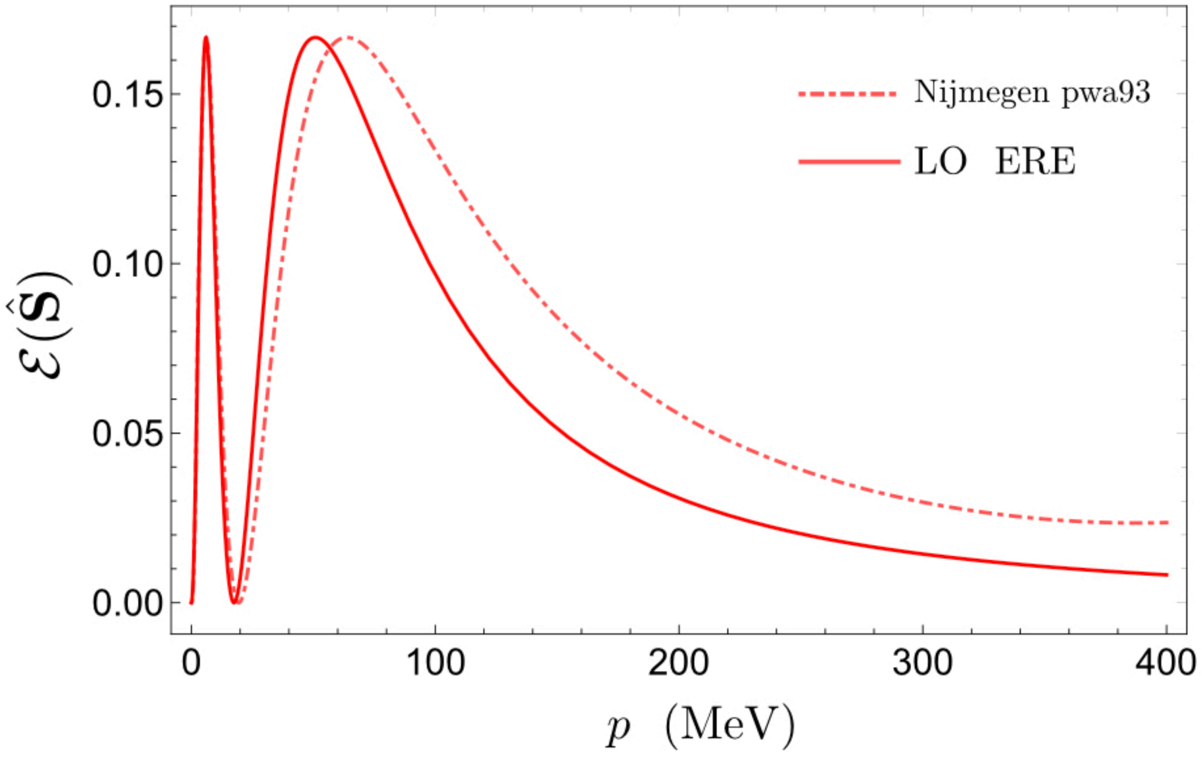
Equi-potential surfaces



$$\mathbb{V}(\phi,\theta) = \frac{|a_0 a_1|}{(|a_0|+|a_1|)^2\, c_1^2}\tan^2\left(\tfrac{1}{2}(\phi+\epsilon\,\theta)\right)$$

Equi-entanglement surfaces





UV/IR symmetry!

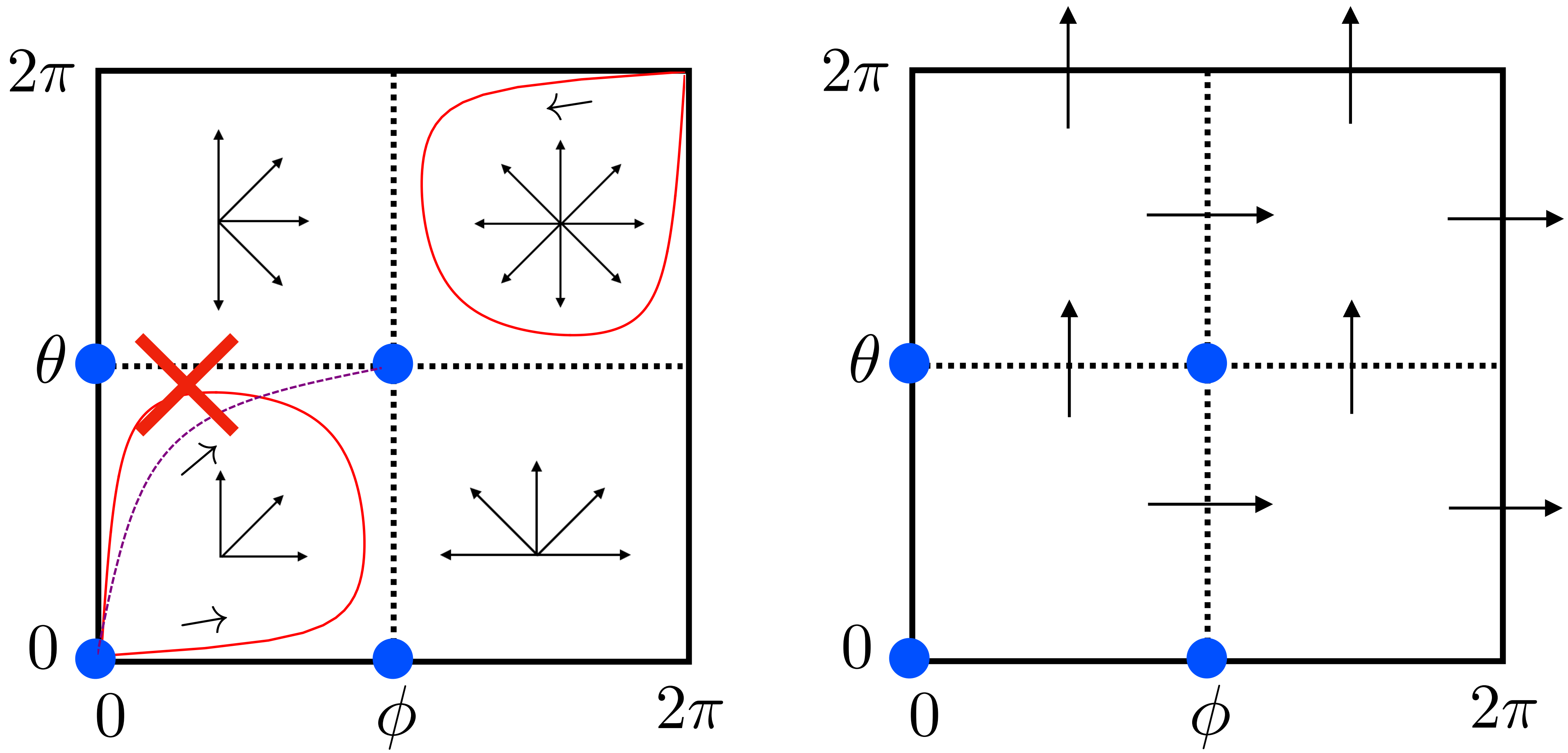
$$p \mapsto \frac{1}{|a_1 a_0| p}$$

$$\mathcal{E}(\hat{\mathbf{S}}) = N_{\mathcal{P}} \sin^2(\phi - \theta)$$

Causality on the flat torus (Wigner bounds)

$$r \;\leq\; 2\left[\mathbf{R} - \frac{\mathbf{R}^2}{a} + \frac{\mathbf{R}^3}{3a^2}\right]$$

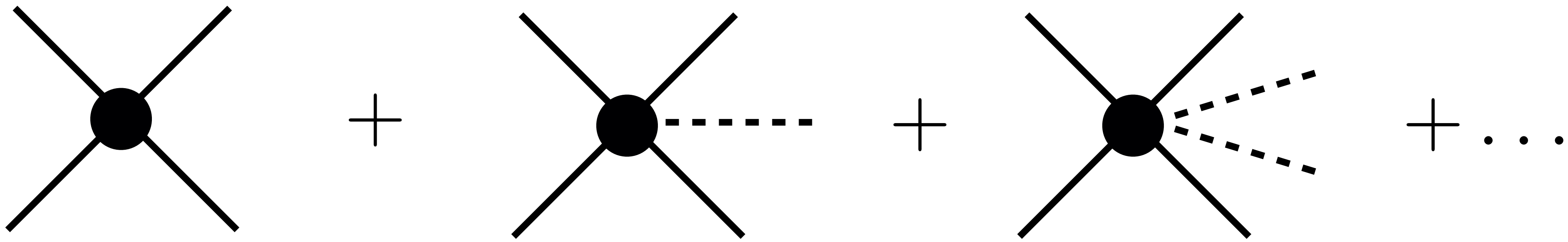
Causal trajectories are geometrically constrained



$$\dot{\phi}(p) \geq \frac{\sin\phi(p)}{p} \quad , \quad \dot{\theta}(p) \geq \frac{\sin\theta(p)}{p}$$

Inelasticity and holography

Now allow inelastic loss



$$\hat{\mathbf{S}}_I^\dagger \hat{\mathbf{S}}_I = \hat{\mathbf{S}}^\dagger \hat{\mathbf{S}} - \sum_\gamma |\gamma\rangle\langle\gamma|$$

Assumption: both channels couple to single source

$$\exp 2i\delta_0 \to \eta_0 \exp 2i\delta_0$$

$$\exp 2i\delta_1 \to \eta_1 \exp 2i\delta_1$$

$$\Longrightarrow$$

$$SU(4)_W$$

$$\eta_0 = \eta_1 = r$$

Embedding in four-dimensional Euclidean space $\mathbb{R}^4$

$$ds^2 \;=\; dr^2 \;+\; \tfrac{1}{2}\, r^2 \left(d\phi^2 \;+\; d\theta^2\right)$$

This is a hyperbolic space with curvature and Einstein tensor

$$R \;=\; -\frac{2}{r^2} \qquad\qquad G_{ij} \;=\; \frac{1}{r^2}\delta_i^1\delta_j^1$$

There is a singularity at the maximal violation of unitarity

The trajectory equations are:

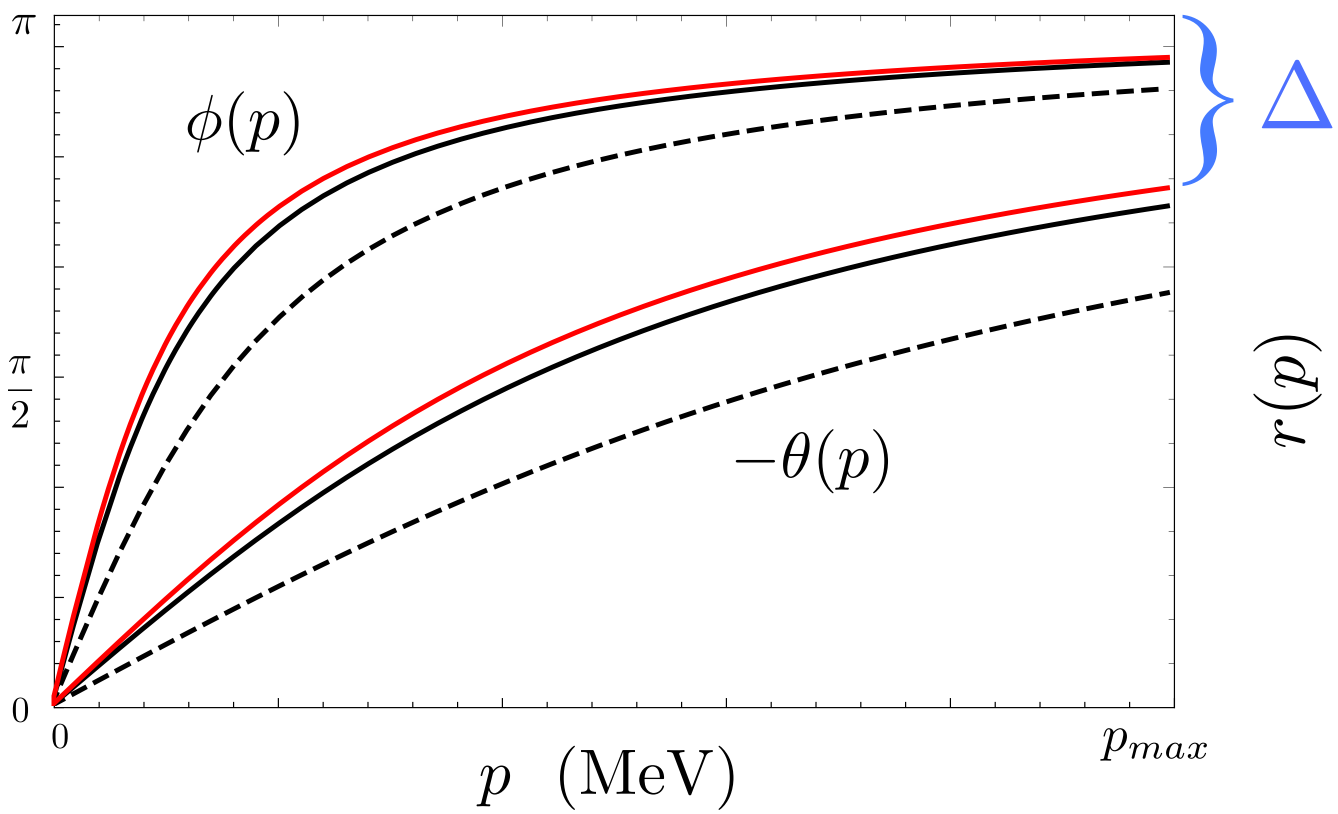
$$\begin{aligned}
\ddot{r} &= \hat{\kappa}(p)\dot{r} \;+\; \tfrac{1}{2} r[(\dot{\phi})^2 \;+\; (\dot{\theta})^2] \;-\; \tfrac{1}{2}\hat{N}^2\partial_r\hat{\mathbb{V}} \\
\ddot{\phi} &= \hat{\kappa}(p)\dot{\phi} \;-\; 2\dot{\phi}\,\frac{\dot{r}}{r} \;-\; \hat{N}^2\frac{1}{r^2}\partial_\phi\hat{\mathbb{V}} \\
\ddot{\theta} &= \hat{\kappa}(p)\dot{\theta} \;-\; 2\dot{\theta}\,\frac{\dot{r}}{r} \;-\; \hat{N}^2\frac{1}{r^2}\partial_\theta\hat{\mathbb{V}}
\end{aligned}$$

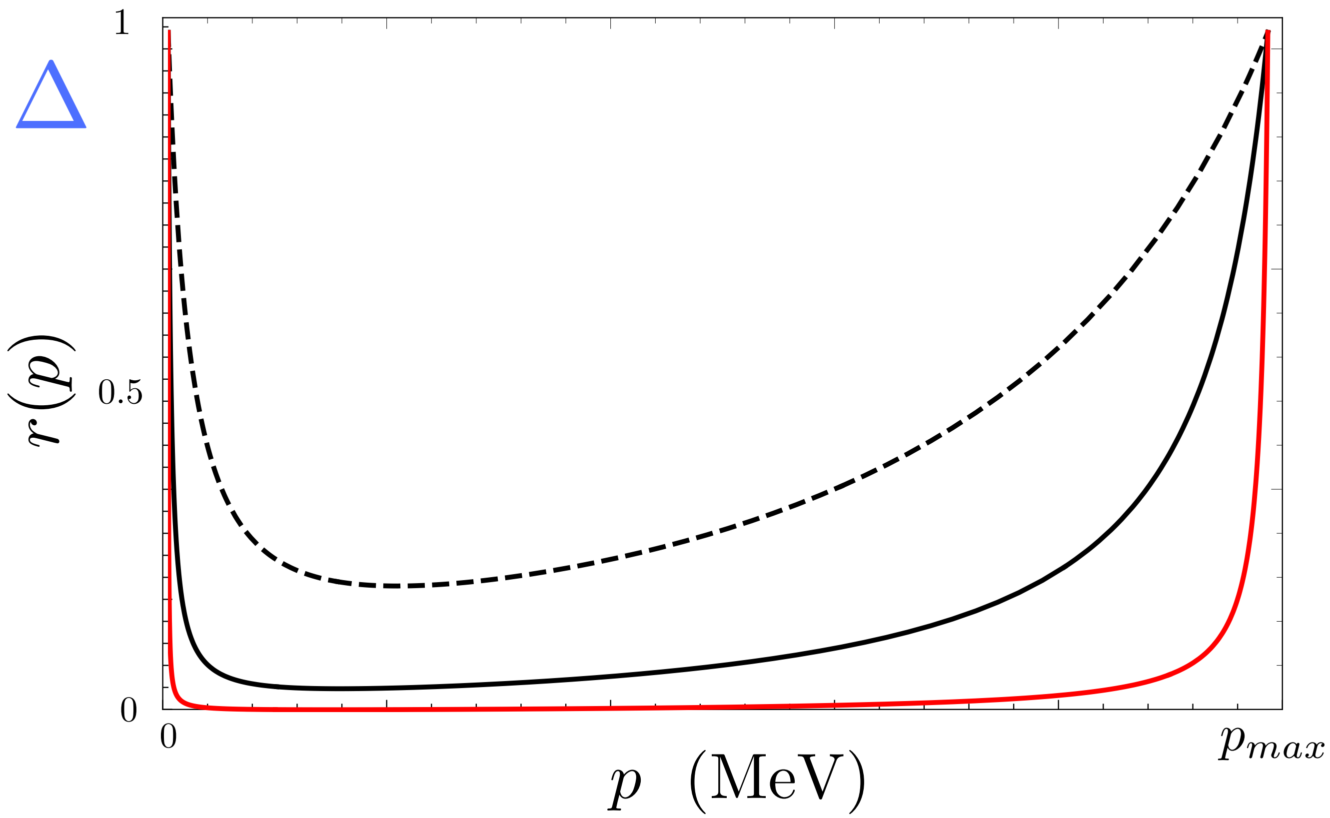
Is it possible to engineer a bulk potential which will reproduce LO in the effective range expansion while allowing for inelastic lossiness?

Yes, but with an intrinsic error

$$\hat{\mathbb{V}}(r,\phi,\theta) = \frac{1}{r^2}\mathbb{V}(\phi,\theta) \;=\; \frac{|a_0 a_1|}{\left(|a_0|+|a_1|\right)^2}\,\frac{1}{r^2}\;\tan^2\left(\tfrac{1}{2}(\phi+\epsilon\,\theta)\right)$$

$$r(p) \;=\; \cos\left(\mathcal{A}\tfrac{1}{2}(\phi_{\max}-\epsilon\,\theta_{\max})\right)\sec\left(\mathcal{A}\left[(\phi(p)-\epsilon\,\theta(p)) \;-\; \tfrac{1}{2}(\phi_{\max}-\epsilon\,\theta_{\max})\right]\right)$$



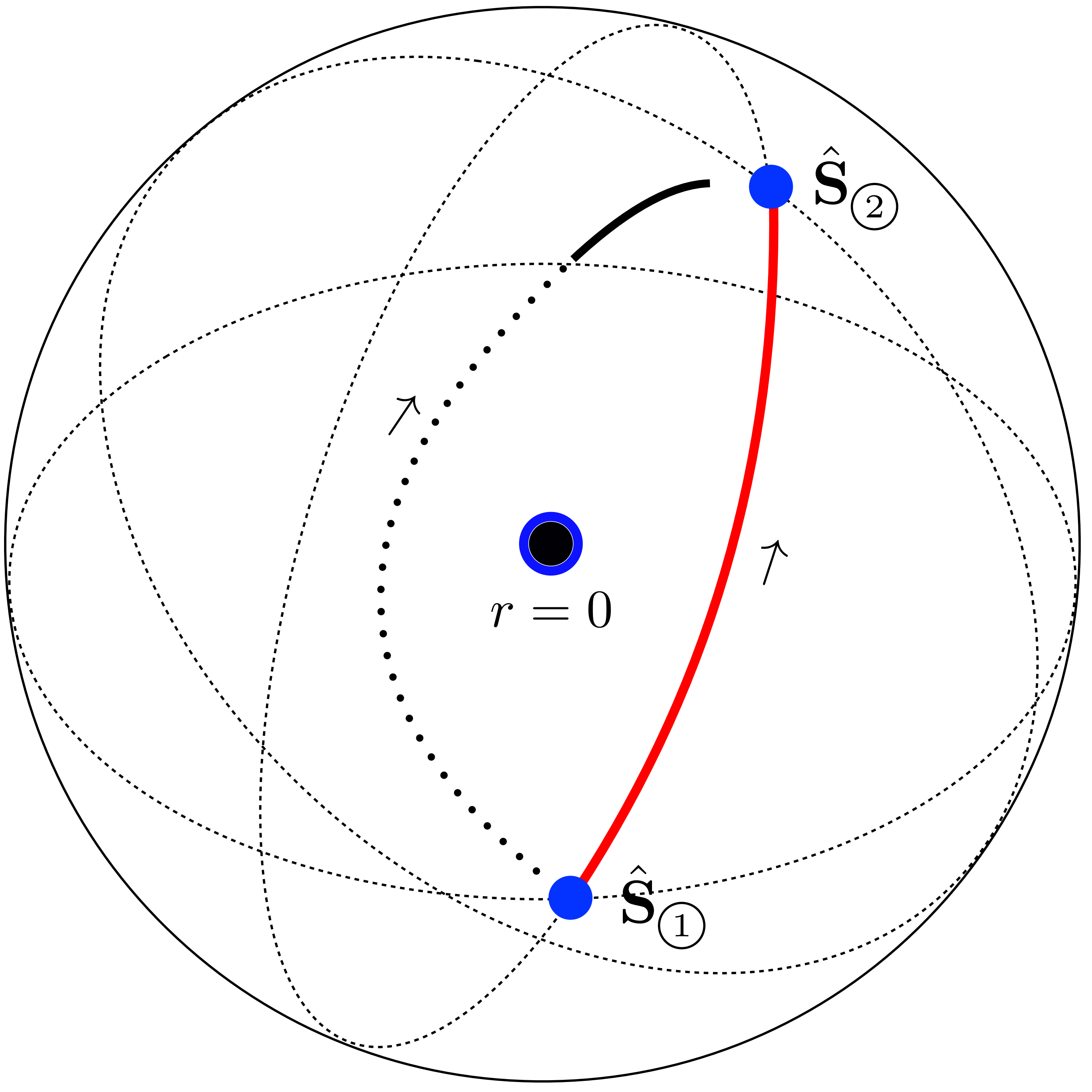


$$\phi = \pi \;+\; \frac{2}{a_0 p_{\max}} + \mathcal{O}\left((a_0 p_{\max})^{-3}\right) \;=\; 0.94\,\pi$$

$$\theta = -\pi \;+\; \frac{2}{a_1 p_{\max}} + \mathcal{O}\left((a_1 p_{\max})^{-3}\right) \;=\; -0.75\,\pi$$

$$p_{\max} < 89.4\ \text{MeV}$$

The flat torus is the boundary of the hyperbolic space



In usual holographic duality, CFT on the boundary is unitary. Here the boundary is in some sense unitarity itself.

Summary

- In baryon-baryon scattering, minimization of the EP implies new symmetries in the strange sector which simplify the effective field theory. Lattice QCD simulations agree with these predictions. Entanglement constrains scattering.

- Fermion-Fermion (qubit-qubit) scattering has a geometric formulation in which the S-matrix propagates in a theory space generated by entanglement and bounded by unitarity. With inelastic loss, it is a simple toy model of holography.

- Minimization of the EP in pion-pion (qutrit-qutrit) and pion -nucleon (qubit-qutrit) scattering leads to consequences that are indistinguishable from large-N QCD implications.

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