4 Finite-range T matrix calculation

4.1 Problem description

4.2 Integral form in 3D coordinate basis

The T matrix takes the form:

$$T_{\beta\alpha}^{M_a M_A M_b M_B} = \langle \phi_B^{M_B} \chi_b^{M_b(-)} | V_{\text{post/prior}} | \chi_a^{M_A(+)} \phi_a^{M_a} \rangle \tag{36}$$

expand it into an integral form:

$$\begin{aligned} T_{\beta\alpha}^{M_a M_A M_b M_B} = & \int d\vec{r}_a d\vec{r}_{bx} d\vec{r}_b d\vec{r}_x \langle \phi_B^{M_B} \chi_b^{M_b(-)} | \vec{r}_b \vec{r}_x \rangle \langle \vec{r}_b \vec{r}_x | V_{\text{post/prior}} | \vec{r}_a \vec{r}_{bx} \rangle \\ & \times \langle \vec{r}_a \vec{r}_{bx} | \chi_a^{M_A(+)} \phi_a^{M_a} \rangle \end{aligned} \tag{37}$$

ignore the spin of the relative motion scattering wave function:

$$\begin{aligned} T_{\beta\alpha}^{M_a M_B} = & \int d\vec{r}_a d\vec{r}_{bx} d\vec{r}_b d\vec{r}_x \langle \phi_B^{M_B} \chi_b^{(-)} | \vec{r}_b \vec{r}_x \rangle \langle \vec{r}_b \vec{r}_x | V_{\text{post/prior}} | \vec{r}_a \vec{r}_{bx} \rangle \\ & \times \langle \vec{r}_a \vec{r}_{bx} | \chi_a^{(+)} \phi_a^{M_a} \rangle \end{aligned} \tag{38}$$

the four inner products can be expressed with:

$$\langle \chi_b^{(-)} | \vec{r}_b \rangle = \frac{4\pi}{k_b r_b} \sum_{l_b \ m_b} i^{l_b} (-1)^{l_b} u_{l_b}(r_b) e^{i\sigma_{l_b}} Y_{l_b}^{m_b}(\hat{r}_b) Y_{l_b}^{m_b *}(\hat{k}_b) \tag{39}$$

$$\langle \phi_B^{M_B} | \vec{r}_x \rangle = \frac{u_{l_x}(r_x)}{r_x} Y_{l_x}^{M_B *}(\hat{r}_x) \tag{40}$$

$$\langle \vec{r}_a | \chi_a^{(+)} \rangle = \frac{4\pi}{k_a r_a} \sum_{l_a m_a} i^{l_a} u_{l_a}(r_a) e^{i\sigma_{l_a}} Y_{l_a}^{m_a}(\hat{r}_a) Y_{l_a}^{m_a *}(\hat{k}_a) \tag{41}$$

$$\langle \vec{r}_{bx} | \phi_a^{M_a} \rangle = \frac{u_{l_{bx}}(r_{bx})}{r_{bx}} Y_{l_{bx}}^{M_a}(\hat{r}_{bx}) \tag{42}$$

the complex conjugation should be taken seriously.

The post/prior potential takes the local form:

$$\langle \vec{r}_b \vec{r}_x | V_{\text{post/prior}} | \vec{r}_a \vec{r}_{bx} \rangle = V_{\text{post/prior}}(r_{bx}, r_x, x) \delta(\vec{f}(\vec{r}_{bx}, \vec{r}_x) - \vec{r}_a) \delta(\vec{g}(\vec{r}_{bx}, \vec{r}_x) - \vec{r}_b) \tag{43}$$

where x stands for the angle between r_{bx} and r_x.

Combine all the expressions:

$$\begin{aligned} T_{\beta\alpha}^{M_a M_B} = & \frac{16\pi^2}{k_a k_b} \int d\vec{r}_{bx} d\vec{r}_x \frac{u_{l_x}(r_x)}{r_x} \frac{u_{l_{bx}}(r_{bx})}{r_{bx}} Y_{l_{bx}}^{M_a}(\hat{r}_{bx}) Y_{l_x}^{M_B *}(\hat{r}_x) \\ & \times V_{\text{post/prior}}(r_{bx} r_x x) \sum_{l_a l_b} i^{l_a + l_b} (-1)^{l_b} e^{i(\sigma_{l_a} + \sigma_{l_b})} \frac{u_{l_a}(f)}{f} \frac{u_{l_b}(g)}{g} \\ & \times \sum_{m_a m_b} Y_{l_a}^{m_a}(\hat{f}) Y_{l_a}^{m_a *}(\hat{k}_a) Y_{l_b}^{m_b}(\hat{g}) Y_{l_b}^{m_b *}(\hat{k}_b) \end{aligned} \tag{44}$$

4.3 Manipulation of spherical harmonics

In order to carry out the numerical integration, we need to unify the variables. With the expansion of spherical harmonics:

$$\begin{aligned} Y_l^m(\hat{\boldsymbol{r}}) =& \sqrt{4\pi}\sum_{n=0}^{l}\sum_{\lambda=-n}^{n} c(l,n)\frac{(pR_1)^{l-n}(qR_2)^n}{r^l} \\ & \times Y_{l-n}^{m-\lambda}(\hat{\boldsymbol{R}}_{\boldsymbol{1}})Y_n^{\lambda}(\hat{\boldsymbol{R}}_{\boldsymbol{2}})\langle l-n\ m-\lambda, n\lambda|lm\rangle \end{aligned} \tag{45}$$

where

$$c(l,n) = \left(\frac{(2l+1)!}{(2n+1)![2(l-n)+1]!}\right)^{1/2} \tag{46}$$

when the vector satisfies:

$$\vec{r} = p\vec{R_1} + q\vec{R_2} \tag{47}$$

so we can use r_{bx} and r_x to express f and g:

$$\vec{r_a} = \vec{f} = \vec{r_x} - p\vec{r}_{bx} \tag{48}$$

$$\vec{r_b} = \vec{g} = q\vec{r_x} - \vec{r}_{bx} \tag{49}$$

where

$$p = \frac{m_b}{m_b + m_x} \tag{50}$$

$$q = \frac{m_A}{m_A + m_x} \tag{51}$$

so the two spherical harmonics of f and g can be expanded with:

$$\begin{aligned} Y_{l_b}^{m_b}(\hat{g}) =& \sqrt{4\pi}\sum_{n=0}^{l_b}\sum_{\lambda=-n}^{n} c(l_b,n)\frac{(qr_x)^{l_b-n}(-r_{bx})^n}{g^{l_b}}Y_{l_b-n}^{m_b-\lambda}(\hat{r_x})Y_n^{\lambda}(\hat{r_{bx}}) \\ & \times \langle l_b-n, n, m_b-\lambda, \lambda \mid l_b, m_b\rangle \end{aligned} \tag{52}$$

$$\begin{aligned} Y_{l_a}^{m_a}\left(\hat{f}\right) =& \sqrt{4\pi}\sum_{u=0}^{l_a}\sum_{\nu=-u}^{u} c(l_a,u)\frac{(-pr_{bx})^{l_a-u}(r_x)^u}{f^{l_a}}Y_{l_a-u}^{m_a-\nu}(\hat{r_{bx}})Y_u^{\nu}(\hat{r_x}) \\ & \times \langle l_a-u, u, m_a-\nu, \nu \mid l_a, m_a\rangle \end{aligned} \tag{53}$$

combine all the spherical harmonics:

$$\begin{aligned} T_{\beta\alpha}^{M_aM_B} =& \frac{64\pi^3}{k_ak_b}\sum_{l_al_b} e^{i(\sigma_{l_a}+\sigma_{l_b})}i^{l_a+l_b}\int dr_{bx}dr_x(-1)^{l_b+n}(-p)^{l_a-u}q^{l_b-n} \\ & \times r_{bx}^{n+l_a+1-u}r_x^{l_b-n+1+u}u_{l_x}(r_x)u_{l_{bx}}(r_{bx}) \\ & \times \sum_{m_am_b}\sum_{nu}\sum_{\lambda\nu} Y_{l_a}^{m_a*}(\hat{k}_a)Y_{l_b}^{m_b*}(\hat{k}_b)c(l_a,u)\,c(l_b,n) \\ & \times \int d\hat{r}_xd\hat{r}_{bx}Y_{l_{bx}}^{M_a}(\hat{r}_{bx})Y_{l_x}^{M_B*}(\hat{r}_x)V_{\text{post/prior}}(r_{bx},r_x,x)\frac{u_{l_a}(f)}{f^{l_a+1}}\frac{u_{l_b}(g)}{g^{l_b+1}} \\ & \times Y_{l_a-u}^{m_a-\nu}(\hat{r_{bx}})Y_u^{\nu}(\hat{r_x})Y_{l_b-n}^{m_b-\lambda}(\hat{r_x})Y_n^{\lambda}(\hat{r_{bx}}) \\ & \times \langle l_a-u, u; m_a-\nu, \nu \mid l_a, m_a\rangle\langle l_b-n, n; m_b-\lambda, \lambda \mid l_b, m_b\rangle \end{aligned} \tag{54}$$

since the length of f and g depends on the relative angle x, in order to separate the angle and the radius variable, we carry out the Legendre expansion:

$$V_{\text{post/prior}}(r_{bx}, r_x, x)\frac{u_{l_a}(f)}{f^{l_a+1}}\frac{u_{l_b}(g)}{g^{l_b+1}} = \sum_{T=0}^{T_{\max}}(2T+1)\mathbf{q}^T_{l_a,l_b}(r_{bx}, r_x)\, P_T(x) \tag{55}$$

where:

$$\mathbf{q}^T_{l_a,l_b}(r_{bx}, r_x) = \frac{1}{2}\int_{-1}^{1} V_{\text{post/prior}}(r_{bx}, r_x, x)\frac{u_{l_a}(f)}{f^{l_a+1}}\frac{u_{l_b}(g)}{g^{l_b+1}}P_T(x)\mathrm{d}x \tag{56}$$

and the Legendre function can be represented with spherical harmonics as well through the so-called addition theorem:

$$P_T(x) = \frac{4\pi}{2T+1}\sum_{m_T=-T}^{T}(-1)^{m_T}Y_T^{-m_T}(\hat{r}_{bx})\, Y_T^{m_T}(\hat{r_x}) \tag{57}$$

all the angle part now can be represented with spherical harmonics:

$$\begin{aligned} T^{M_a M_B}_{\beta\alpha} =& \frac{(4\pi)^4}{k_a k_b}\sum_{l_a l_b} e^{i(\sigma_{l_a}+\sigma_{l_b})} i^{l_a+l_b}\int dr_{bx}dr_x(-1)^{l_b+n}(-p)^{l_a-u}q^{l_b-n} \\ &\times r_{bx}^{n+l_a+1-u} r_x^{l_b-n+1+u} u_{l_x}(r_x)u_{l_{bx}}(r_{bx}) \\ &\times \sum_{m_a m_b}\sum_{nu}\sum_{\lambda\nu}\sum_{T m_T} Y_{l_a}^{m_a *}(\hat{k}_a)Y_{l_b}^{m_b *}(\hat{k}_b)c\,(l_a, u)\, c\,(l_b, n)\,\mathbf{q}^T_{l_a,l_b}(r_{bx}, r_x) \\ &\times \int d\hat{r}_x Y_u^{\nu}(\hat{r_x})\, Y_{l_b-n}^{m_b-\lambda}(\hat{r_x})\, Y_{l_x}^{M_B *}(\hat{r}_x)Y_T^{m_T}(\hat{r_x}) \\ &\times \int d\hat{r}_{bx} Y_n^{\lambda}(\hat{r_{bx}})\, Y_{l_a-u}^{m_a-\nu}(\hat{r_{bx}})\, Y_{l_{bx}}^{M_a}(\hat{r}_{bx})(-1)^{m_T}Y_T^{-m_T}(\hat{r}_{bx}) \\ &\times \langle l_a-u, u, m_a-\nu, \nu \mid l_a, m_a\rangle\,\langle l_b-n, n, m_b-\lambda, \lambda \mid l_b, m_b\rangle \end{aligned} \tag{58}$$

the multiplication of two spherical harmonics can be reduced with:

$$\begin{aligned} Y_{l_1,m_1}(\theta,\phi)Y_{l_2,m_2}(\theta,\phi) = &\sum_{L=|l_1-l_2|}^{l_1+l_2}\sum_{M=-L}^{+L}(-1)^M \times \left[\frac{(2l_1+1)(2l_2+1)(2L+1)}{4\pi}\right]^{1/2} \\ &\times \begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & M \end{pmatrix} Y_{L,-M}(\theta,\phi) \end{aligned} \tag{59}$$

so the first two terms of two angles can be reduced with:

$$\begin{aligned} Y_u^{\nu}(\hat{r_x})\, Y_{l_b-n}^{m_b-\lambda}(\hat{r_x}) =& \sum_{\Lambda_b}\sum_{m_{\Lambda_b}}(-)^{m_{\Lambda_b}}\left[\frac{(2l_b-2n+1)(2u+1)(2\Lambda_b+1)}{4\pi}\right]^{1/2} \\ &\times \begin{pmatrix} u & l_b-n & \Lambda_b \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} u & l_b-n & \Lambda_b \\ \nu & m_b-\lambda & -m_{\Lambda_b} \end{pmatrix} Y_{\Lambda_b}^{m_{\Lambda_b}}(\hat{r_x}) \end{aligned} \tag{60}$$

$$\begin{aligned} Y_n^{\lambda}(\hat{r}_{bx})\, Y_{l_a-u}^{m_a-\nu}(\hat{r}_{bx}) =& \sum_{\Lambda_a}\sum_{m_{\Lambda_a}}(-)^{m_{\Lambda_a}}\left[\frac{(2l_a-2u+1)(2n+1)(2\Lambda_a+1)}{4\pi}\right]^{1/2} \\ &\times \begin{pmatrix} n & l_a-u & \Lambda_a \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} n & l_a-u & \Lambda_a \\ \lambda & m_a-\nu & -m_{\Lambda_a} \end{pmatrix} Y_{\Lambda_a}^{m_a}(\hat{r}_{bx}) \end{aligned} \tag{61}$$

$2\times 4=8$ spherical harmonics now are reduced to $2\times 3=6$ ones. It is worthy to mention that the integral is carried out only on these spherical harmonics, and the three harmonics integral can be represented with 3j symbol:

$$\begin{aligned}&\int Y_{l_1,m_1}(\theta,\phi)Y_{l_2,m_2}(\theta,\phi)Y_{l_3,m_3}(\theta,\phi)\mathrm{d}\Omega\\=&\left[\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}\right]^{1/2}\begin{pmatrix}l_1&l_2&l_3\\0&0&0\end{pmatrix}\begin{pmatrix}l_1&l_2&l_3\\m_1&m_2&m_3\end{pmatrix}\end{aligned}\tag{62}$$

so the remaining integration on 6 harmonics can be expressed with:

$$\begin{aligned}&\int Y_{\Lambda_a}^{m_{\Lambda_a}}(\hat{r}_{bx})\,Y_{l_{bx}}^{m_{bx}}(\hat{r}_{bx})\,Y_{T}^{-m_T}(\hat{r}_{bx})\,\mathrm{d}\Omega\\=&\left[\frac{(2\Lambda_a+1)(2l_{bx}+1)(2T+1)}{4\pi}\right]^{1/2}\begin{pmatrix}\Lambda_a&l_{bx}&T\\0&0&0\end{pmatrix}\begin{pmatrix}\Lambda_a&l_{bx}&T\\m_{\Lambda_a}&m_{bx}&-m_T\end{pmatrix}\end{aligned}\tag{63}$$

$$\begin{aligned}&\int Y_{\Lambda_b}^{m_{\Lambda_b}}(\hat{r}_{x})\,Y_{l_{x}}^{M_B*}(\hat{r}_{x})\,Y_{T}^{m_T}(\hat{r}_{x})\,\mathrm{d}\Omega\\=&(-)^{M_B}\left[\frac{(2\Lambda_b+1)(2l_{x}+1)(2T+1)}{4\pi}\right]^{1/2}\begin{pmatrix}\Lambda_b&l_{x}&T\\0&0&0\end{pmatrix}\begin{pmatrix}\Lambda_b&l_{x}&T\\m_{\Lambda_b}&-M_B&m_T\end{pmatrix}\end{aligned}\tag{64}$$

then convert the last two CG coefficients to 3j symbol with:

$$\langle j_1 j_2; m_1 m_2 | j_1 j_2; J\ M\rangle=(-1)^{j_1-j_2+M}\sqrt{2J+1}\begin{pmatrix}j_1&j_2&J\\m_1&m_2&-M\end{pmatrix}\tag{65}$$

which reads:

$$\langle l_a-u,u;m_a-\nu,\nu\mid l_a,m_a\rangle=(-)^{l_a+m_a}\sqrt{2l_a+1}\begin{pmatrix}l_a-u&u&l_a\\m_a-\nu&\nu&-m_a\end{pmatrix}\tag{66}$$

$$\langle l_b-n,n;m_b-\lambda,\lambda\mid l_b,m_b\rangle=(-)^{l_b+m_b}\sqrt{2l_b+1}\begin{pmatrix}l_b-n&n&l_b\\m_b-\lambda&\lambda&-m_b\end{pmatrix}\tag{67}$$

we use the convention to simplify the expression:

$$\hat{n}=\sqrt{2n+1}\tag{68}$$

finally all the harmonics are replaced by the 3j symbols:

$$
\begin{aligned}
T_{\beta\alpha}^{M_a M_B} =& \frac{(4\pi)^4}{k_a k_b} \sum_{l_a l_b T} e^{i(\sigma_{l_a}+\sigma_{l_b})} i^{l_a+l_b} \int dr_{bx} dr_x (-1)^{l_a+n} (-p)^{l_a-u} q^{l_b-n} \\
& \times r_{bx}^{n+l_a+1-u} r_x^{l_b-n+1+u} u_{l_x}(r_x) u_{l_{bx}}(r_{bx}) \mathbf{q}_{l_a,l_b}^T(r_{bx}, r_x) \\
& \times \sum_{m_a m_b} \sum_{nu} Y_{l_a}^{m_a*}(\hat{k}_a) Y_{l_b}^{m_b*}(\hat{k}_b) c(l_a, u)\, c(l_b, n) \\
& \times \sum_{\Lambda_a \Lambda_b} \hat{l_a} \hat{l_b} \hat{l_x} \hat{l}_{bx} \hat{n} \hat{u} \widehat{(l_b-n)} \widehat{(l_a-u)} \hat{\Lambda}_b^2 \hat{\Lambda}_a^2 \hat{T}^2 \begin{pmatrix} \Lambda_a & l_{bx} & T \\ 0 & 0 & 0 \end{pmatrix} \\
& \times \begin{pmatrix} \Lambda_b & l_x & T \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n & l_a-u & \Lambda_a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u & l_b-n & \Lambda_b \\ 0 & 0 & 0 \end{pmatrix} \\
& \times \sum_{\lambda\nu m_T} \sum_{m_{\Lambda_a} m_{\Lambda_b}} (-1)^{m_T+m_{\Lambda_a}+m_{\Lambda_b}+M_B+m_a+m_b} \\
& \times \begin{pmatrix} \Lambda_a & l_{bx} & T \\ m_{\Lambda_a} & m_{bx} & -m_T \end{pmatrix} \begin{pmatrix} \Lambda_b & l_x & T \\ m_{\Lambda_b} & -M_B & m_T \end{pmatrix} \begin{pmatrix} n & l_a-u & \Lambda_a \\ \lambda & m_a-\nu & -m_{\Lambda_a} \end{pmatrix} \\
& \times \begin{pmatrix} u & l_b-n & \Lambda_b \\ \nu & m_b-\lambda & -m_{\Lambda_b} \end{pmatrix} \begin{pmatrix} l_a-u & u & l_a \\ m_a-\nu & \nu & -m_a \end{pmatrix} \begin{pmatrix} l_b-n & n & l_b \\ m_b-\lambda & \lambda & -m_b \end{pmatrix}
\end{aligned} \tag{69}
$$

then we have to deal with these complicated 3j symbols.

4.4 Reduction of the 3j symbol

In this section, we focus on the sum over the last 6 3j symbols. With the reduction:

$$
\begin{aligned}
& \sum_b (2b+1) \begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} b & e & h \\ \beta & \epsilon & \eta \end{pmatrix} \begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix} \\
=& \sum_{\phi\nu\delta\rho} \begin{pmatrix} c & f & i \\ \gamma & \phi & v \end{pmatrix} \begin{pmatrix} a & d & g \\ \alpha & \delta & \rho \end{pmatrix} \begin{pmatrix} d & e & f \\ \delta & \epsilon & \phi \end{pmatrix} \begin{pmatrix} g & h & i \\ \rho & \eta & \nu \end{pmatrix}
\end{aligned} \tag{70}
$$

the last 4 symbols can be reduced to:

$$
\begin{aligned}
& \sum_{\lambda\nu} \begin{pmatrix} n & l_a-u & \Lambda_a \\ \lambda & m_a-\nu & -m_{\Lambda_a} \end{pmatrix} \begin{pmatrix} u & l_b-n & \Lambda_b \\ \nu & m_b-\lambda & -m_{\Lambda_b} \end{pmatrix} \\
& \times \begin{pmatrix} l_a-u & u & l_a \\ m_a-\nu & \nu & -m_a \end{pmatrix} \begin{pmatrix} l_b-n & n & l_b \\ m_b-\lambda & \lambda & -m_b \end{pmatrix} \\
=& (-)^{n+\Lambda_a+l_a-u} \sum_l (2l+1) \begin{pmatrix} l_b & l & l_a \\ -m_b & m_l & -m_a \end{pmatrix} \\
& \times \begin{pmatrix} l & \Lambda_a & \Lambda_b \\ m_l & -m_{\Lambda_a} & -m_{\Lambda_b} \end{pmatrix} \begin{Bmatrix} l_b & l & l_a \\ n & \Lambda_a & l_a-u \\ l_b-n & \Lambda_b & u \end{Bmatrix}
\end{aligned} \tag{71}
$$

so the sum over the 6 3j symbols turned into:

$$\begin{aligned}
&\sum_{m_T}\sum_{m_{\Lambda_a}m_{\Lambda_b}}(-1)^{m_T+m_{\Lambda_a}+m_{\Lambda_b}+M_B+m_a+m_b+n+\Lambda_a+l_a-u}\\
&\times\begin{pmatrix}\Lambda_a & l_{bx} & T\\ m_{\Lambda_a} & m_{bx} & -m_T\end{pmatrix}\begin{pmatrix}\Lambda_b & l_x & T\\ m_{\Lambda_b} & -M_B & m_T\end{pmatrix}\\
&\times\sum_l \hat{l}^2\begin{pmatrix}l_b & l & l_a\\ -m_b & m_l & -m_a\end{pmatrix}\begin{pmatrix}l & \Lambda_a & \Lambda_b\\ m_l & -m_{\Lambda_a} & -m_{\Lambda_b}\end{pmatrix}\begin{Bmatrix}l_b & l & l_a\\ n & \Lambda_a & l_a-u\\ l_b-n & \Lambda_b & u\end{Bmatrix}
\end{aligned}\tag{72}$$

sum over 3 symbols can be reduced with:

$$\begin{aligned}
&W(abcd;ef)\begin{pmatrix}c & a & f\\ \gamma & \alpha & \phi\end{pmatrix}\\
=&\sum_{\beta\delta\varepsilon}(-)^{f-e-\alpha-\delta}\begin{pmatrix}a & b & e\\ \alpha & \beta & -\epsilon\end{pmatrix}\begin{pmatrix}d & c & e\\ \delta & \gamma & \epsilon\end{pmatrix}\begin{pmatrix}b & d & f\\ \beta & \delta & -\phi\end{pmatrix}
\end{aligned}\tag{73}$$

so the sum over 3 3j symbols can be converted to:

$$\begin{aligned}
&\sum_{m_T}\sum_{m_{\Lambda_a}m_{\Lambda_b}}(-1)^{m_{\Lambda_a}+M_B}\begin{pmatrix}\Lambda_a & l_{bx} & T\\ m_{\Lambda_a} & m_{bx} & -m_T\end{pmatrix}\\
&\times\begin{pmatrix}\Lambda_b & l_x & T\\ m_{\Lambda_b} & -M_B & m_T\end{pmatrix}\begin{pmatrix}l & \Lambda_a & \Lambda_b\\ m_l & -m_{\Lambda_a} & -m_{\Lambda_b}\end{pmatrix}\\
=&(-)^{l_x+\Lambda_a}W\left(l_x,\Lambda_b,l_{bx},\Lambda_a;T,l\right)\begin{pmatrix}l_{bx} & l_x & l\\ m_{bx} & -M_B & -m_l\end{pmatrix}
\end{aligned}\tag{74}$$

then the sum over 6 3j symbols turned into:

$$\begin{aligned}
&\sum_l(2l+1)(-)^{m_T+l_x+m_{\Lambda_b}+m_a+m_b+n+l_a-u}W\left(l_x,\Lambda_b,l_{bx},\Lambda_a;T,l\right)\\
&\times\begin{pmatrix}\Lambda_b & l_x & T\\ m_{\Lambda_b} & -M_B & m_T\end{pmatrix}\begin{pmatrix}l_{bx} & l_x & l\\ m_{bx} & -M_B & -m_l\end{pmatrix}\begin{Bmatrix}l_b & l & l_a\\ n & \Lambda_a & l_a-u\\ l_b-n & \Lambda_b & u\end{Bmatrix}
\end{aligned}\tag{75}$$

at last we arrive at the final compact form of T matrix:

$$
\begin{aligned}
T_{\beta\alpha}^{M_a M_B} =& \frac{(4\pi)^4}{k_a k_b} \sum_{l_a l_b T} e^{i(\sigma_{l_a}+\sigma_{l_b})} i^{l_a+l_b} \int dr_{bx} dr_x (-1)^{l_a+n} (-p)^{l_a-u} q^{l_b-n} \\
& \times r_{bx}^{n+l_a+1-u} r_x^{l_b-n+1+u} u_{l_x}(r_x) u_{l_{bx}}(r_{bx}) \mathbf{q}_{l_a,l_b}^{T}(r_{bx}, r_x) \\
& \times \sum_{m_a m_b} \sum_{nu} Y_{l_a}^{m_a *}(\hat{k}_a) Y_{l_b}^{m_b *}(\hat{k}_b) c\,(l_a, u)\, c\,(l_b, n) \\
& \times \sum_{\Lambda_a \Lambda_b} \hat{l}_a \hat{l}_b \hat{l}_x \hat{l}_{bx} \hat{n} \hat{u} (\widehat{l_b - n})(\widehat{l_a - u}) \hat{\Lambda}_b^2 \hat{\Lambda}_a^2 \hat{T}^2 \begin{pmatrix} \Lambda_a & l_{bx} & T \\ 0 & 0 & 0 \end{pmatrix} \\
& \times \begin{pmatrix} \Lambda_b & l_x & T \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n & l_a - u & \Lambda_a \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u & l_b - n & \Lambda_b \\ 0 & 0 & 0 \end{pmatrix} \\
& \times \sum_{l} (2l+1) (-)^{m_T + l_x + m_{\Lambda_b} + m_a + m_b + n + l_a - u} W\,(l_x, \Lambda_b, l_{bx}, \Lambda_a; T, l) \\
& \times \begin{pmatrix} \Lambda_b & l_x & T \\ m_{\Lambda_b} & -M_B & m_T \end{pmatrix} \begin{pmatrix} l_{bx} & l_x & l \\ m_{bx} & -M_B & -m_l \end{pmatrix} \begin{Bmatrix} l_b & l & l_a \\ n & \Lambda_a & l_a - u \\ l_b - n & \Lambda_b & u \end{Bmatrix}
\end{aligned}
\tag{76}
$$