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Comparative study of three body models for continuum states

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Outline of this paper

Hyperspherical Harmonics (HH) basis

1, R-matrix method

2, Complex scaling method

3, Pseudostate method

Application: 6He, phase shift, dipole transition

$$\alpha + n + n$$

Introduction of HH basis

Define the Jacobi coordinate:

$$\boldsymbol{x}_{N-j+1} = \sqrt{\frac{2m_{j+1}M_j}{M_{j+1}m}}\left(\boldsymbol{r}_{j+1} - \boldsymbol{X}_j\right) \qquad \boldsymbol{X}_j = \frac{1}{M_j}\sum_{i=1}^{j} m_i \boldsymbol{r}_i$$

Define the hyperradius:

$$\rho = \sqrt{\sum_{i=1}^{N} x_i^2} = \sqrt{\frac{2}{A}\sum_{j>i=1}^{A}\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right)^2} = \sqrt{2\sum_{i=1}^{A}\left(\boldsymbol{r}_i - X\right)^2},$$

Define the hyperangle: $$\cos\varphi_j = \frac{x_j}{\sqrt{x_1^2 + \ldots + x_j^2}}$$

3B situation: $$\rho = \sqrt{\boldsymbol{x}_k^2 + \boldsymbol{y}_k^2}, \quad \tan\alpha_k = y_k/x_k$$

Introduction of HH basis

All the angle variable: $\Omega_{5k} = \left(\alpha_k, \Omega_x, \Omega_y\right)$

T operator in CM frame:

Grand angular momentum Operator

$$T_\rho = -\frac{\hbar^2}{2m_N}\left(\frac{\partial^2}{\partial\rho^2} + \frac{5}{\rho}\frac{\partial}{\partial\rho} - \frac{K^2\left(\Omega_5\right)}{\rho^2}\right)$$

Eignefuntion of the operator is the HH function:

$$K^2\mathcal{Y}^{JM}_{\gamma K} = K(K+4)\mathcal{Y}^{JM}_{\gamma K}$$

The index contains a set of quantum numbers:

$$\gamma = \left(\ell_x, \ell_y, L, S\right)$$

Introduction of HH basis

The HH function has the form:

$$\mathscr{Y}_{\gamma K}^{JM}\left(\Omega_5\right)=\phi_K^{\ell_x\ell_y}(\alpha)\left[\mathscr{Y}_{\ell_x\ell_y}^{L}\otimes\chi^S\right]^{JM}$$

$$\mathscr{Y}_{\ell_x\ell_y}^{LM_L}\left(\Omega_5\right)=\left[Y_{\ell_x}\left(\Omega_x\right)\otimes Y_{\ell_y}\left(\Omega_y\right)\right]^{LM_L}$$

The wave function can be expanded in:

$$\Psi^{JM\pi}=\rho^{-5/2}\sum_{K=0}^{\infty}\sum_{\gamma}\chi_{\gamma K}^{J\pi}(\rho)\mathscr{Y}_{\gamma K}^{JM}\left(\Omega_5\right)=\rho^{-5/2}\sum_{K=0}^{\infty}\sum_{\gamma}\Psi_{\gamma K}^{JM\pi}$$

The Schrödinger equation can be written in:

$$\left(-\frac{\hbar^2}{2m_N}\left[\frac{d^2}{d\rho^2}-\frac{\mathscr{L}_K\left(\mathscr{L}_K+1\right)}{\rho^2}\right]-E\right)\chi_{\gamma K}^{J\pi}(\rho)+\sum_{K'\gamma'}V_{\gamma K,\gamma' K'}^{J\pi}(\rho)\chi_{\gamma' K'}^{J\pi}(\rho)=0$$

Introduction of HH basis

The potential matrix is given by:

$$V^{J\pi}_{\gamma K,\gamma' K'}(\rho) = \left\langle \mathcal{Y}^{JM}_{\gamma K} \middle| \sum_{i>j} V_{ij}\left(\boldsymbol{r}_i - \boldsymbol{r}_j\right) \middle| \mathcal{Y}^{JM}_{\gamma' K'} \right\rangle$$

The potential takes the form:

$$V^{J\pi}_{\gamma K,\gamma' K'}(\rho) \underset{\rho\to\infty}{\longrightarrow} \delta_{LL'}\delta_{SS'} \sum_{k=0}^{\infty} \frac{v_k}{\rho^{2k+3}}$$

Projection over the forbidden states:

$$V_{ij} \to V_{ij} + \Lambda \sum_n \left|\varphi_n\right\rangle \left\langle\varphi_n\right|$$

Dipole strength

The E1 transition strength from gs to a continuum is

$$\frac{dB(E1)}{dE} = \frac{1}{2J_0+1} \sum_{\gamma_\omega K_\omega} \sum_{JM\pi M_0 \mu} \left| \left\langle \mathscr{K} \Psi^{JM\pi}_{\gamma_\omega K_\omega}(E) \middle| \mathscr{M}^{E1}_{\mu} \middle| \Psi^{J_0 M_0 \pi_0} \right\rangle \right|^2 ,$$

The dipole electric operator takes the form(HH form)

$$\mathscr{M}^{\mathrm{E1}}_{\mu} = eZ_c \left(\frac{2}{A_c A} \right)^{1/2} \rho \sin \alpha Y^{\mu}_{1} \left(\Omega_y \right)$$

The matrix element can be calculated:

$$\left\langle \Psi^{J\pi}_{\gamma K} \left\| \mathscr{M}^{E\lambda} \right\| \Psi^{J'\pi'}_{\gamma' K'} \right\rangle = eZ_c \left(\frac{2}{A_c A} \right)^{\lambda/2} C^{JJ'}_{\gamma K \gamma' K'}$$

$$\times \int_0^\infty \chi^{J\pi}_{\gamma K}(\rho) \rho^\lambda \chi^{J'\pi'}_{\gamma' K'}(\rho) d\rho \int_0^{\frac{\pi}{2}} \phi^{\ell_x \ell_y}_{K}(\alpha) \phi^{\ell'_\ell \ell'_y}_{K'}(\alpha) (\cos\alpha)^2 (\sin\alpha)^{\lambda+2} d\alpha \, .$$

R-matrix method

The radial function has the asymptotic form:

$$\chi_{\gamma K}^{J\pi}(\rho) \underset{\rho\to\infty}{\longrightarrow} \chi_{\gamma K,\text{ as}}^{J\pi}(\rho),$$

$$\chi_{\gamma K,\text{ as}}^{J\pi}(\rho) = i^{K_\omega+1}(k/4\pi E)^{1/2}\left[H_{\gamma K}^{-}(k\rho)\delta_{\gamma\gamma_\omega}\delta_{KK_\omega} - U_{\gamma K,\gamma_\omega K_\omega}^{J\pi}H_{\gamma K}^{+}(k\rho)\right]$$

The wave function in the internal region can be expanded:

$$\chi_{\gamma K,\text{int}}^{J\pi}(\rho) = \sum_{i=1}^{N} c_{\gamma K i}^{J\pi}u_i(\rho)$$

Introduce the Bloch-Schrödinger equation:

$$(H + \mathscr{L} - E)\Psi^{JM\pi} = \mathscr{L}\Psi^{JM\pi},$$

The R-matrix can be expressed as:

$$R_{\gamma K,\gamma' K'}^{J\pi}\left(a_0\right) = \frac{\hbar^2}{2m_N a_0}\sum_{i,i'} u_i\left(a_0\right)\left(\boldsymbol{C}^{J\pi}\right)_{\gamma K i,\gamma' K' i'}^{-1} u_{i'}\left(a_0\right),$$

CSM

We are solving the complexed scaled Schrödinger equation:

$$H(\theta)\Psi(\theta) = U(\theta)HU^{-1}(\theta)\Psi(\theta) = E(\theta)\Psi(\theta)$$

The complex eigenvalues are:

$$E(\theta) = E_r - i\Gamma_r/2$$

The response function can be expressed :

$$R_{E1}\left(J_0\pi_0 \rightarrow J\pi, E\right) = \sum_{\lambda} \frac{\left\langle \tilde{\Psi}_{\lambda}^{J\pi}(\theta) \left\| \mathscr{M}_{\theta}^{E1} \right\| \Psi^{J_0\pi_0}(\theta) \right\rangle^2}{E - E_{\lambda}^{J}(\theta)}$$

The dipole strength is obtained from the response function:

$$\frac{dB(E1)}{dE} = -\frac{1}{\pi} \operatorname{Im} R_{E1}\left(J_0\pi_0 \rightarrow J\pi, E\right)$$

pseudostate method

The pseudostate method just solve the eigenvalue problem:

$$H\Psi_{\lambda}^{JM\pi} = E_{\lambda}^{J\pi}\Psi_{\lambda}^{JM\pi}$$

The dipole strength for specific energy is obtained with:

$$B_{E1}\left(J_0\pi_0 \to J\pi, E_{\lambda}^{J\pi}\right) = \frac{2J+1}{2J_0+1}\left|\left\langle \Psi_{\lambda}^{J\pi}\left(E_{\lambda}^{J\pi}\right) || \mathscr{M}^{E1} || \Psi^{J_0\pi_0}\right\rangle\right|^2$$

The differential relation can be recovered by:

$$\frac{dB(E1)}{dE} \approx \sum_{J\pi\lambda} f\left(E, E_{\lambda}^{J\pi}\right) B_{E1}\left(J_0\pi_0 \to J\pi, E_{\lambda}^{J\pi}\right)$$

The function is in general chosen as a Gaussian, with a width parameter.

Application to 6He

For exotic nuclei, the binding energy is very small, and many observables are very sensitive to its value.

Fine-tuning is required for its potential

1. Renormalizing the $\alpha+n$ by a factor $\lambda_{\alpha n}$. This factor is close to unity (see Table I), but slightly affects the $\alpha+n$ phase shifts. In particular the $p_{3/2}$ and $p_{1/2}$ resonances are lower than the experimental values.
2. Rescaling the attractive part of the $n+n$ Minnesota interaction.[31)] This option leaves the $\alpha+n$ phase shifts unchanged.
3. Introducing a phenomenological three-body potential depending on the hyperradius only. This potential is defined as

$$V_3(\rho) = \frac{v_3}{1+\exp\left(\rho/\rho_3\right)}\delta_{KK'}\delta_{\gamma\gamma'},$$

3B phase shift

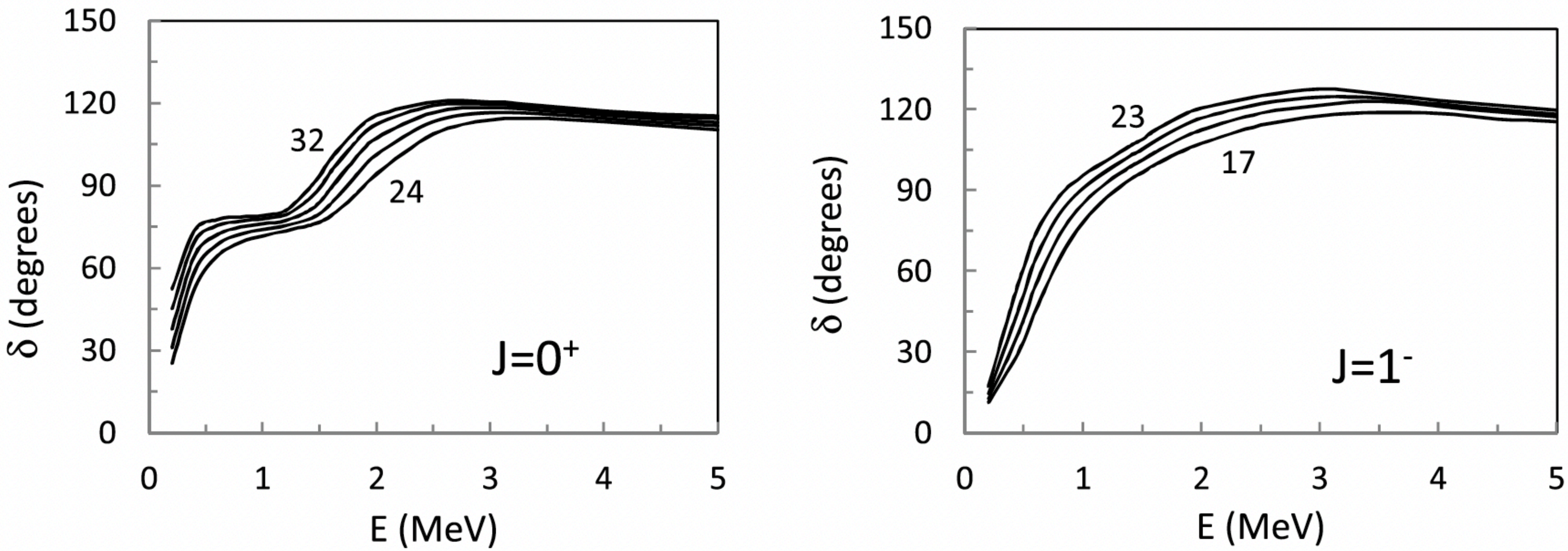

Fig. 1. $\alpha + n + n$ eigenphase shifts for $J = 0^+$ (left panel) and $J = 1^-$ (right panel), and for the renormalized $n + n$ potential. The $K_{\max}$ values vary by steps of 2 (the minimum and maximum values are indicated).

3B phase shift

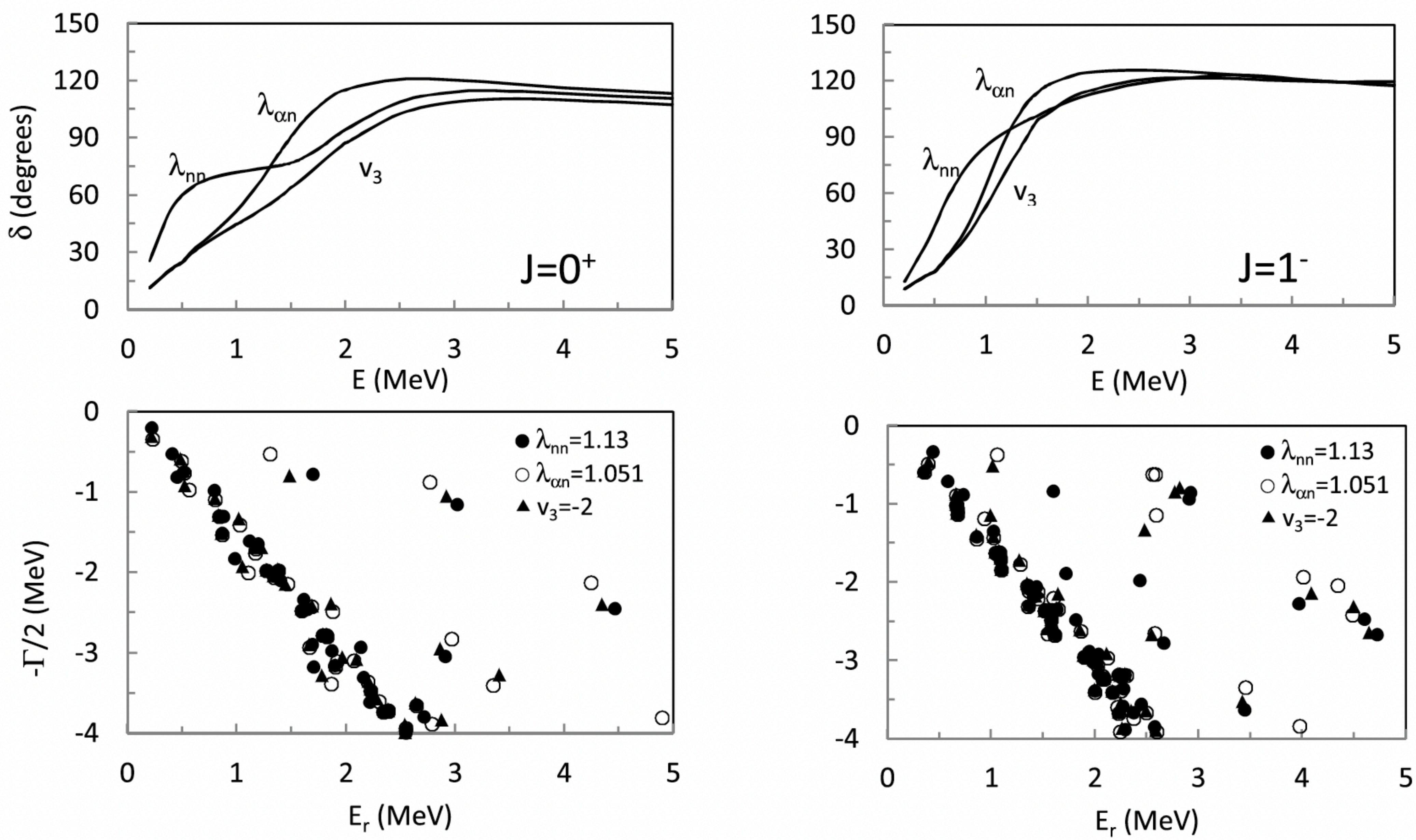

Fig. 2. Upper panel: 0^+ and 1^- eigenphase shifts for different conditions of calculations ($K_{\max} = 24$ for $J = 0^+$ and $K_{\max} = 19$ for $J = 1^-$. Lower panel: corresponding CSM energies with $\theta = 0.5$.

Dipole strength

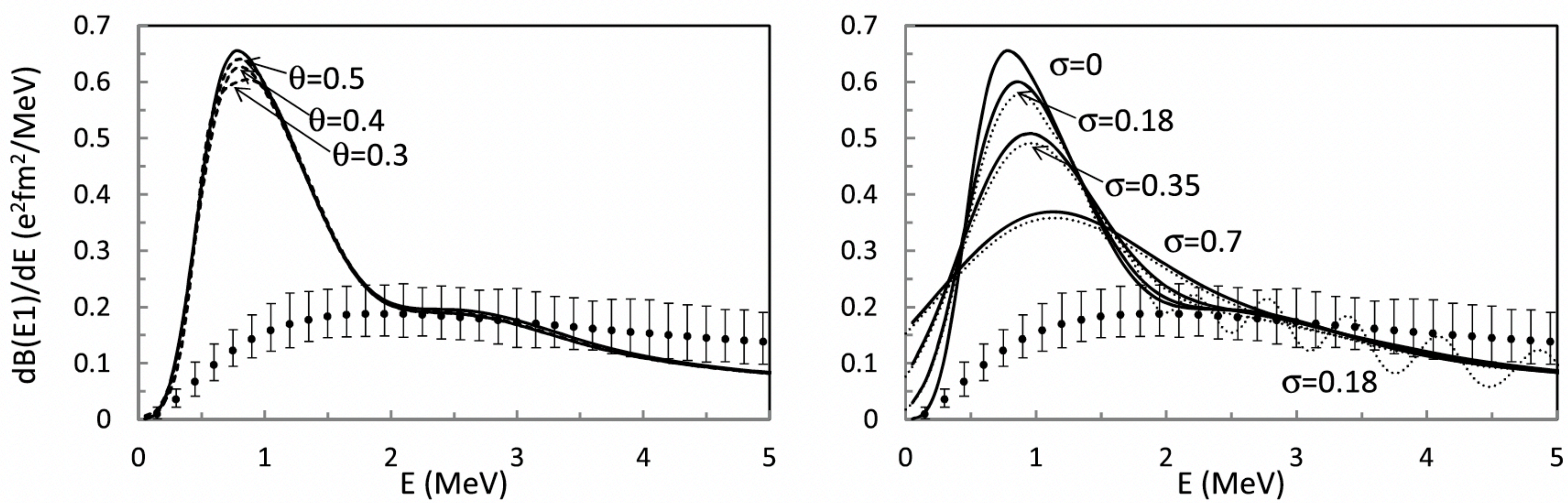

Fig. 3. ^{6}He dipole strengths. Left panel: comparison of the R-matrix (solid lines) and CSM (dashed lines) methods for different θ values. Right panel: comparison of the R-matrix (solid lines) and PS (dotted lines) methods for different σ values. The experimental data are taken from Ref. 36).

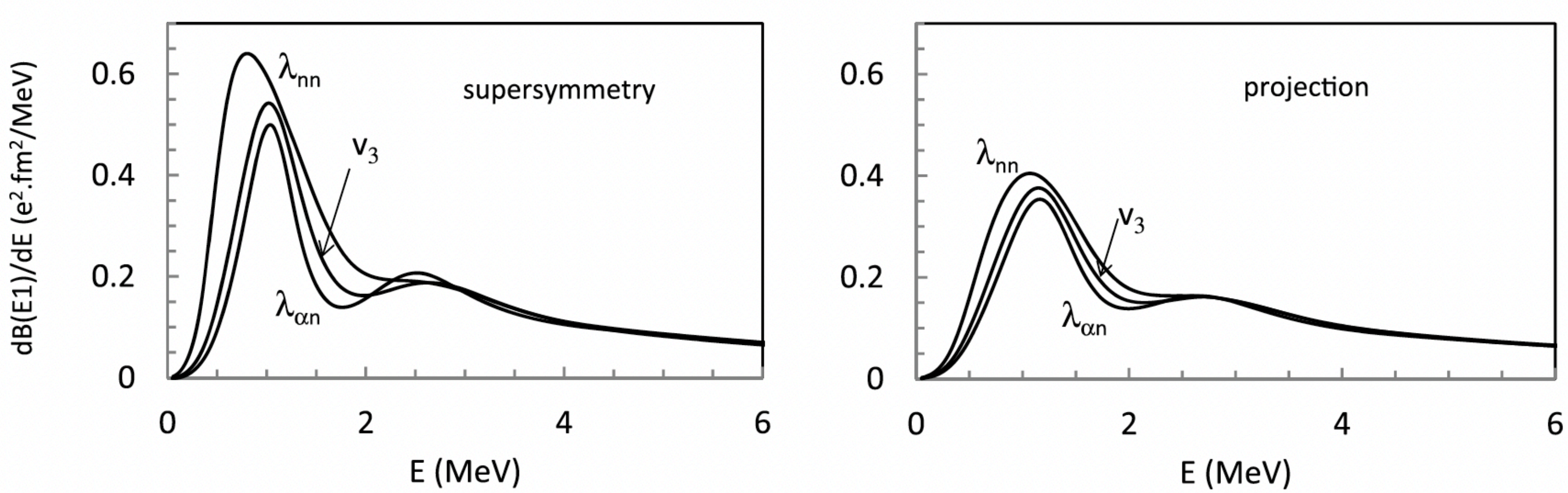

Fig. 4. ^{6}He dipole-strength distributions with different potentials (see Table I) and with two methods for removing $\alpha + n$ forbidden states: supersymmetry (left panel) and projection (right panel).