Tensor product of state spaces

Definition and properties

Tensor product space

$$\varepsilon = \varepsilon_1 \otimes \varepsilon_2$$

if there is associated with each pair of vectors, $|\psi(1)\rangle \in \varepsilon_1$ and $|\psi(2)\rangle \in \varepsilon_2$, a vector of ε , denoted by $|\psi(1)\rangle \otimes |\psi(2)\rangle$.

1. It is **linear** with respect to multiplication by complex numbers.
2. It is **distributive** with respect to vector addition.
3. The set of basis $|u_i(1)\rangle \otimes |v_l(2)\rangle$ constitutes a basis in ε. The dimension of ε is $N_1 N_2$.

$$\langle \varphi'(1)\chi'(2)|\varphi(1)\chi(2)\rangle = \langle \varphi'(1)|\varphi(1)\rangle \langle \chi'(2)|\chi(2)\rangle$$
$$\langle u_{i'}(1)v_{l'}(2)|u_i(1)v_l(2)\rangle = \langle u_{i'}(1)|u_i(1)\rangle \langle v_{l'}(2)|v_l(2)\rangle = \delta_{ii'}\delta_{ll'}$$

Tensor product of operators

The extension of a linear operator $A(1)$ in ε :

$$\tilde{A}(1)|\psi\rangle = \sum_{i,l} c_{i,l}[A(1)|u_i(1)\rangle] \otimes |v_l(2)\rangle$$

It is easy to show that two operators such as $\tilde{A}(1)$ and $\tilde{B}(2)$ commute in ε.

The tensor product $A(1) \otimes B(2)$ is the linear operator in ε.

$$[A(1) \otimes B(2)][|\psi(1)\rangle \otimes |\chi(2)\rangle] = [A(1)|\psi(1)\rangle] \otimes [B(2|\chi(2)\rangle)]$$

Eigenvalue equations in the product space

Eigenvalue equation of $A(1)$

$$A(1)|\varphi_n^i(1)\chi(2)\rangle = [A(1)|\varphi_n^i(1)\rangle] \otimes |\chi(2)\rangle = a_n|\varphi_n^i(1)\chi(2)\rangle$$

If $A(1)$ is an observable in ε_1, the orthonormal system of vectors $\left|\psi_n^{i,l}\right\rangle$ is a basis in ε :

$$\left|\psi_n^{i,l}\right\rangle = \left|\varphi_n^i(1)\right\rangle \otimes |v_l(2)\rangle$$

Eigenvalue equation of $A(1)+B(2)$

$$C = A(1) + B(2)$$
$$C\,|\varphi_n(1)\chi_p(2)\rangle = (a_n + b_p)\,|\varphi_n(1)\chi_p(2)\rangle$$

The eigenvalues of C are the sums of an eigenvalue of $A(1)$ and an eigenvalue of $B(2)$. One can find a basis of eigenvectors of which are tensor products of an eigenvector of $A(1)$ and an eigenvector of $B(2)$.

> ***Comment:***
>
> Equation (F-30) shows that the eigenvalues of C are all of the form $c_{np} = a_n + b_p$. If two different pairs of values of n and p which give the same value for c_{np} do not exist, c_{np} is non-degenerate (recall that we have assumed a_n and b_p to be non-degenerate in $\mathscr{E}_1$ and $\mathscr{E}_2$ respectively). The corresponding eigenvector of C is necessarily the tensor product $|\,\varphi_n(1)\,\rangle\,|\,\chi_p(2)\,\rangle$. If, on the other hand, the eigenvalue c_{np} is, for example, two-fold degenerate (there exist m and q such that $c_{mq} = c_{np}$), all that can be asserted is that every eigenvector of C corresponding to this eigenvalue is written:
>
> $$\lambda\,|\,\varphi_n(1)\,\rangle\,|\,\chi_p(2)\,\rangle + \mu\,|\,\varphi_n(1)\,\rangle\,|\,\chi_p(2)\,\rangle \qquad \text{(F-31)}$$
>
> where λ and μ are arbitrary complex numbers. In this case, therefore, there exist eigenvectors of C which are not tensor products.

$$C = \tilde{A}(1) + \tilde{B}(2) = A(1)\otimes \mathbb{I}(2) + \mathbb{I}(1)\otimes B(2)$$

Complete sets of commuting observables in ε

By joining two sets of commuting observables which are complete in ε_1 and ε_2 respectively, one obtains a complete set of commuting observables in ε .

Applications

System of three spin $1/2$ particles

$$\varepsilon_S = \varepsilon_S(1)\otimes\varepsilon_S(2)\otimes\varepsilon_S(3)$$
$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$$

$$S_z = \tilde{S}_{1z} + \tilde{S}_{2z} + \tilde{S}_{3z}, \quad \text{with}$$
$$\tilde{S}_{iz}\,|\varepsilon_1\varepsilon_2\varepsilon_3\rangle = \frac{\hbar}{2}\varepsilon_i\,|\varepsilon_1\varepsilon_2\varepsilon_3\rangle, \quad i = 1,2,3$$

eigenvalues : $3/2\hbar,\ 1/2\hbar\times 3,\ -1/2\hbar\times 3,\ -3/2\hbar$

```
u1 = [1,0]; u2 = [0,1]    > Vector{Int64} with 2 elements
Sz1 = 1/2 * [1 0 ; 0 -1]    > 2×2 Matrix{Float64}:
Sz = kron(Sz1,I(2),I(2)) + kron(I(2),Sz1,I(2)) + kron(I(2),I(2),Sz1)    > 8×8
Sz * kron(u2,u2,u2)    > Vector{Float64} with 8 elements
eigen(Sz)    > Eigen{Float64, Float64, Matrix{Float64}, Vector{Float64}}
```

$$\begin{aligned} S^2 &= S_1^2 + S_2^2 + S_3^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2 + 2\mathbf{S}_1 \cdot \mathbf{S}_3 + 2\mathbf{S}_2 \cdot \mathbf{S}_3 \\ &= \frac{9}{4}\hbar^2 + 2(\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_3) \end{aligned}$$

$$S^2 \left|\varepsilon_1\varepsilon_2\varepsilon_3\right\rangle = \frac{9}{4}\hbar^2 + \frac{\hbar^2}{2}(\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2 + \sigma_z^1\sigma_z^2 + \cdots) \left|\varepsilon_1\varepsilon_2\varepsilon_3\right\rangle$$

eigenvalues : $3/4\hbar^2 \times 4,\ 15/4\hbar^2 \times 4$

```
σxyz = [ [0 1 ; 1 0], [0 -1im ; 1im 0], [1 0 ; 0 -1] ]   > Vecto
σ12 = [ kron(σ,I(2),I(2)) * kron(I(2),σ,I(2)) for σ in σxyz ]
σ13 = [ kron(σ,I(2),I(2)) * kron(I(2),I(2),σ) for σ in σxyz ]
σ23 = [ kron(I(2),σ,I(2)) * kron(I(2),I(2),σ) for σ in σxyz ]
S2 = 1/2 * (sum(σ12) + sum(σ13) +sum(σ23))   > 8×8 Matrix{Comple
eigen(S2).values .+ 9/4   > Vector{Float64} with 8 elements
```

ECOC : $\{S_{1z}, S_{2z}, S_{3z}\} \to \{S^2, S_z\}$

```
eigen(Sz).vectors   ⌄ 8×8 Matrix{Float64}:
                      0.0  0.0  0.0  0.0  0.0  0.0  0.0  1.0
                      0.0  0.0  0.0  0.0  1.0  0.0  0.0  0.0
                      0.0  0.0  0.0  0.0  0.0  1.0  0.0  0.0
                      0.0  1.0  0.0  0.0  0.0  0.0  0.0  0.0
                      0.0  0.0  0.0  0.0  0.0  0.0  1.0  0.0
                      0.0  0.0  1.0  0.0  0.0  0.0  0.0  0.0
                      0.0  0.0  0.0  1.0  0.0  0.0  0.0  0.0
                      1.0  0.0  0.0  0.0  0.0  0.0  0.0  0.0
```

```
eigen(S2).vectors   ⌄ 8×8 Matrix{ComplexF64}:
                          0.0+0.0im          0.0+0.0im          0.0+0.0im  …  1.0+0.0im
                     0.707107+0.0im          0.0+0.0im  -0.408248+0.0im     0.0+0.0im
                    -0.707107+0.0im          0.0+0.0im  -0.408248+0.0im     0.0+0.0im
                          0.0+0.0im  -0.707107+0.0im          0.0+0.0im     0.0+0.0im
                          0.0+0.0im          0.0+0.0im   0.816497+0.0im     0.0+0.0im
                          0.0+0.0im   0.707107+0.0im          0.0+0.0im  …  0.0+0.0im
                          0.0+0.0im          0.0+0.0im          0.0+0.0im     0.0+0.0im
                          0.0+0.0im          0.0+0.0im          0.0+0.0im     0.0+0.0im
```

Addition of an orbital angular momentum $l = 1$ and a spin $1/2$

$$\begin{gathered} \varepsilon_J = \varepsilon_L \otimes \varepsilon_S \\ \mathbf{J} = \mathbf{L} + \mathbf{S} \end{gathered}$$

$$J_z = \tilde{L}_z + \tilde{S}_z$$

$$\begin{aligned} J^2 &= L^2 + S^2 + 2\mathbf{L} \cdot \mathbf{S} \\ &= l(l+1)\hbar^2 + \frac{3}{4}\hbar^2 + 2(L_x \otimes S_x + L_y \otimes S_y + L_z \otimes S_z) \end{aligned}$$

```
Sxyz = 1/2 * [ [0 1 ; 1 0], [0 -1im ; 1im 0], [1 0 ; 0 -1] ]        > Vector{Matrix{Comp
Lxyz = 1/√2 * [ [0 1 0 ; 1 0 1 ; 0 1 0], 1im*[0 -1 0 ; 1 0 -1 ; 0 1 0],
              √2*[1 0 0 ; 0 0 0 ; 0 0 -1] ]        > Vector{Matrix} with 3 elements
Jz = kron(Lxyz[3],I(2)) + kron(I(3),Sxyz[3])        > 6×6 Matrix{ComplexF64}:
eigen(Jz)        > Eigen{ComplexF64, ComplexF64, Matrix{ComplexF64}, Vector{ComplexF64}}
J2 = 2* sum( [kron(Sxyz[n],Lxyz[n]) for n in 1:3] )        > 6×6 Matrix{ComplexF64}:
eigen(J2).values .+ 11/4        ⌄ Vector{Float64} with 6 elements
                                  0.750…
                                  0.750…
                                  3.75
                                  3.75
                                  3.75
                                  3.75
```

```
eigen(J2).vectors        ⌄ 6×6 Matrix{ComplexF64}:
           0.0+0.0im        0.0+0.0im        0.0+0.0im  …  1.0+0.0im  0.0+0.0im
       0.57735+0.0im        0.0+0.0im   0.816497+0.0im     0.0+0.0im  0.0+0.0im
           0.0+0.0im  -0.816497+0.0im        0.0+0.0im     0.0+0.0im  0.0+0.0im
     -0.816497+0.0im        0.0+0.0im    0.57735+0.0im     0.0+0.0im  0.0+0.0im
           0.0+0.0im    0.57735-0.0im        0.0+0.0im     0.0+0.0im  0.0+0.0im
           0.0+0.0im        0.0+0.0im        0.0+0.0im  …  0.0+0.0im  1.0+0.0im
```

求解类CG系数 求解J^2, J_z的共同本征态ϕ：空间运动用坐标表象描述，自旋状态用S_z表象描述

$$\text{ECOC}: \left|L^2, L_z, S^2, S_z\right\rangle \rightarrow \left|J^2, L^2, S^2, J_z\right\rangle$$

$$\langle\theta,\varphi|\phi\rangle = \langle\theta,\varphi|l,m\rangle \otimes |s,m_s\rangle = c_1 Y_{lm_1}|+\rangle + c_2 Y_{lm_2}|-\rangle$$

$$\begin{aligned} J^2 &= L^2 + S^2 + 2\mathbf{L}\cdot\mathbf{S} \\ &= l(l+1)\hbar^2 + \frac{3}{4}\hbar^2 + 2(L_x \otimes S_x + L_y \otimes S_y + L_z \otimes S_z) \\ &= l(l+1)\hbar^2 + \frac{3}{4}\hbar^2 + \hbar\begin{pmatrix} L_z & L_x - iL_y \\ L_x + iL_y & -L_z \end{pmatrix} \end{aligned}$$

$$\langle\theta,\varphi|J^2|\phi\rangle = l(l+1)\hbar^2 + \frac{3}{4}\hbar^2\langle\theta,\varphi|\phi\rangle + \hbar^2\begin{pmatrix} [c_1 m_1 + c_2\sqrt{(l-m_1)(l+m_1+1)}]Y_{lm_1} \\ [c_1\sqrt{(l-m_1)(l+m_1+1)} - c_2 m_2]Y_{lm_2} \end{pmatrix}$$