Quantum Computation:
Density Operator

Open quantum systems

All real quantum systems are open.

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Interactions with the environment drive decoherence.

↓

It can be controlled using quantum error correction

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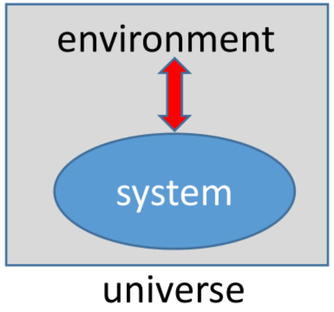
We consider a closed "universe" consisting of a system and its environment.



Axioms

- A state is a ray in Hilbert space.

$$|\psi\rangle \equiv \lambda |\psi\rangle$$

$$a|\phi\rangle + b|\psi\rangle \not\equiv a|\phi\rangle + be^{i\alpha}|\psi\rangle$$

- An observable is a self-adjoint operator on Hilbert space.

$$A = \sum_n a_n E_n \qquad E_n = E_n^\dagger$$

Axioms

- Probabilities of measurement outcomes are determined by the "Born rule".

Post-measurement state:

$$|\psi\rangle \rightarrow \frac{E_n|\psi\rangle}{\|E_n|\psi\rangle\|}$$

Expectation value of the measurement outcome:

$$\mathrm{Prob}(a_n) = \|E_n|\psi\rangle\|^2 = \langle\psi|E_n|\psi\rangle$$

$$\langle A\rangle = \sum_n a_n \mathrm{Prob}(a_n) = \langle\psi|A|\psi\rangle$$

Axioms

- Time evolution is determined by the Schroedinger equation.

  Post-measurement state: $$\frac{d}{dt}|\psi(t)\rangle = -iH(t)|\psi(t)\rangle$$

  The evolution proceeds via a sequence of infinitesimal unitary operators:

  $$\psi(t+dt)\rangle = (I - iH(t)dt)|\psi(t)\rangle = e^{-iH(t)dt}|\psi(t)\rangle = U(t+dt,t)|\psi(t)\rangle$$

- The Hilbert space of composite system AB is the tensor product of A and B:

  $$\mathcal{H}_{AB} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$$

Qubit

$$\dim(\mathcal{H}) = 2, \quad \mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

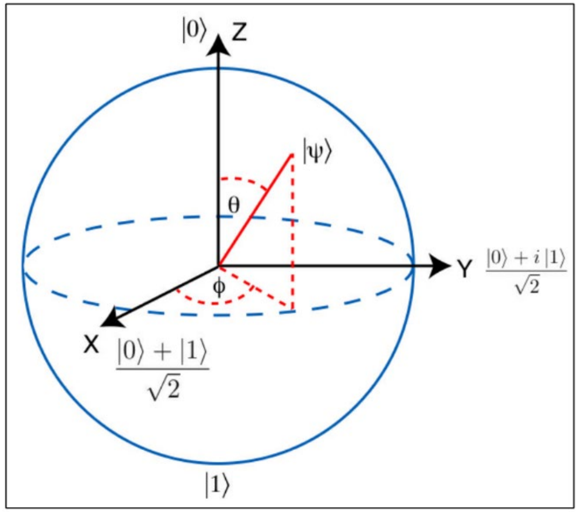
$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$$\text{Prob}(|0\rangle) = |a|^2, \quad \text{Prob}(|1\rangle) = |b|^2, \quad |a|^2 + |b|^2 = 1$$

Qubit

$$|\psi(\theta,\phi)\rangle = e^{-i\phi/2}\cos(\theta/2)|0\rangle + e^{i\phi/2}\sin(\theta/2)|1\rangle$$

$$\theta\in[0,\pi],\quad \phi\in[0,2\pi)$$



$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\hat{n}\cdot\vec{\sigma} = n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3 \rightarrow \hat{n}\cdot\vec{\sigma}\,|\psi(\theta,\phi)\rangle = |\psi(\theta,\phi)\rangle$$

$$\langle\sigma_1\rangle = \sin\theta\cos\phi,\quad \langle\sigma_2\rangle = \sin\theta\sin\phi,\quad \langle\sigma_3\rangle = \cos\theta$$

Quantum "Interference"

Open quantum systems

- States are not rays in Hilbert space.
- Measurements are not orthogonal projections.
- Evolution is not unitary.

Example:

$$|\psi\rangle_{AB} = a|0\rangle_A \otimes |0\rangle_B + b|1\rangle_A \otimes |1\rangle_B = a|00\rangle + b|11\rangle$$

$$A = M_A \otimes I_B \qquad \begin{array}{ll} |0\rangle_A \otimes |0\rangle_B, & \text{Prob} = |a|^2, \\ |1\rangle_A \otimes |1\rangle_B, & \text{Prob} = |b|^2 . \end{array}$$

$$_{AB}\langle\psi| M_A \otimes I_B |\psi\rangle_{AB} = \left(a^*\langle 00| + b^*\langle 11|\right) M_A \otimes I_B \left(a|00\rangle + b|11\rangle\right)$$

$$= |a|^2 \langle 0|M_A|0\rangle + |b|^2 \langle 1|M_A|1\rangle$$

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$${}_{AB}\langle\psi| M_A \otimes I_B |\psi\rangle_{AB} = \mathrm{tr}\left(M_A \rho_A\right)$$

$$\rho_A = |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1|$$

Density operator

Open quantum systems

A more general state of AB:

$$|\psi\rangle_{AB} = \sum_{i,\mu} a_{i\mu} |i\rangle_A \otimes |\mu\rangle_B, \quad \sum_{i,\mu} |a_{i\mu}|^2 = 1$$

$$\begin{aligned} {}_{AB}\langle\psi| M_A \otimes I_B |\psi\rangle_{AB} &= \sum_{j,\nu} a^*_{j\nu} \left({}_A\langle j| \otimes {}_B\langle \nu| \right) \left(M_A \otimes I_B \right) \sum_{i,\mu} a_{i\mu} \left(|i\rangle_A \otimes |\mu\rangle_B \right) \\ &= \sum_{i,j,\mu} a^*_{j\mu} a_{i\mu} \langle j| M_A |i\rangle \end{aligned}$$

$${}_{AB}\langle\psi| M_A \otimes I_B |\psi\rangle_{AB} = \mathrm{tr}\left(\rho_A M_A \right)$$

$$\rho_A = \sum_{i,j,\mu} a^*_{j\mu} a_{i\mu} |i\rangle\langle j| \equiv \mathrm{tr}_B \left(|\psi\rangle\langle\psi| \right)$$

Properties of the density operator

- The density operator is Hermitian.
- The density operator is nonnegative.
- The density operator has unit trace.

$$\rho = \rho^{\dagger}$$

$$\langle \phi | \rho | \phi \rangle = \sum_{i,j,\mu} a_{j\mu}^{*} a_{i\mu} \langle \phi | i \rangle \langle j | \phi \rangle = \sum_{\mu} \left| \sum_{i} a_{i\mu} \langle \phi | i \rangle \right|^2 \geq 0$$

$$\mathrm{tr}\rho = \sum |a_{i\mu}|^2 = \| |\psi\rangle_{AB} \|^2 = 1$$

An orthonormal basis

$$\rho = \sum_{a} p_a |a\rangle\langle a|, \quad p_a \geq 0, \quad \sum_{a} p_a = 1$$

Schmidt decomposition of a bipartite pure state

$$|\psi\rangle_{AB} = \sum a_{i\mu} |i\rangle_A \otimes |\mu\rangle_B, \quad \text{and suppose} \quad \rho_A = \sum_i p_i |i\rangle\langle i|$$

$$|\psi\rangle_{AB} = \sum |i\rangle_A \otimes |\tilde{i}\rangle_B, \quad \text{where} \quad |\tilde{i}\rangle = \sum_\mu a_{i\mu} |\mu\rangle_B$$

$$\rho_A = \sum_i p_i |i\rangle\langle i| = \sum_{i,j} \left(|i\rangle\langle j|\right)_A \mathrm{tr}_B\left(|\tilde{i}\rangle\langle\tilde{j}|\right) = \sum_{i,j} \left(|i\rangle\langle j|\right)_A \langle\tilde{j}|\tilde{i}\rangle$$

$$\langle\tilde{j}|\tilde{i}\rangle = \delta_{ij} p_i \quad \rightarrow \quad |i'\rangle_B = \frac{1}{\sqrt{p_i}} |\tilde{i}\rangle_B \quad (p_i > 0)$$

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i} |i\rangle_A \otimes |i'\rangle_B$$

$$\mathrm{tr}_A\left(|\psi\rangle\langle\psi|\right) = \sum_i p_i |i'\rangle\langle i'|$$

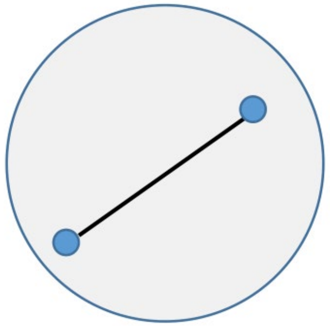
Schmidt decomposition of a bipartite pure state

- The state is separable if and only if there is only one non-zero Schmidt coefficient;
- If more than one Schmidt coefficients are non-zero, then the state is entangled;
- If all the Schmidt coefficients are non-zero and equal, then the state is said to be maximally entangled.

The set of density operators is convex

$$\rho(\lambda) = \lambda\rho_1 + (1-\lambda)\rho_2, \quad 0 \le \lambda \le 1.$$

- Positive. $\langle\psi|\rho(\lambda)|\psi\rangle = \lambda\langle\psi|\rho_1|\psi\rangle + (1-\lambda)\langle\psi|\rho_2|\psi\rangle \ge 0$
- Hermitian.
- Has unit trace.

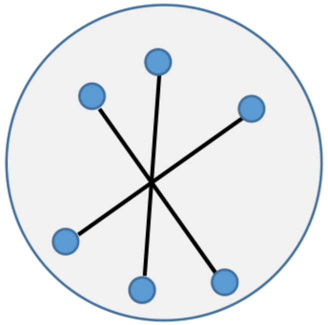


$$\langle M\rangle = \lambda \mathrm{tr}\rho_1 M + (1-\lambda)\mathrm{tr}\rho_2 M = \mathrm{tr}\rho(\lambda)M$$

$\rho = |\psi\rangle\langle\psi| = \lambda\rho_1 + (1-\lambda)\rho_2$ and $\langle\psi^\perp|\psi\rangle = 0.$ Then

$$0 = \langle\psi^\perp|\rho|\psi^\perp\rangle = \lambda\langle\psi^\perp|\rho_1|\psi^\perp\rangle + (1-\lambda)\langle\psi^\perp|\rho_2|\psi^\perp\rangle$$

$$\Rightarrow \langle\psi^\perp|\rho_1|\psi^\perp\rangle = 0 \quad \text{and} \quad \langle\psi^\perp|\rho_2|\psi^\perp\rangle = 0.$$

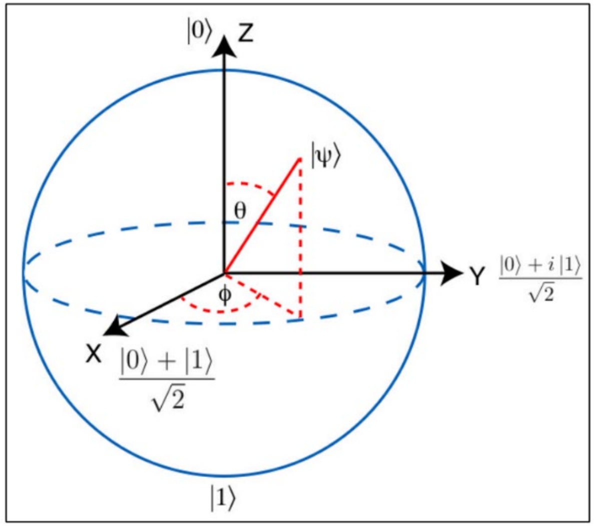


$\rho_1, \rho_2 \propto |\psi\rangle\langle\psi|$ A pure state cannot be obtained as a mixture of two other states

Density operator of a qubit

$$\rho(\vec{P}) = \frac{1}{2}\left(I + \vec{P}\cdot\vec{\sigma}\right) = \frac{1}{2}\begin{pmatrix} 1+P_3 & P_1 - iP_2 \\ P_1 + iP_2 & 1-P_3 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



The Bloch Sphere

$$\det \rho(\vec{P}) = \frac{1}{4}\left(1 - \vec{P}^2\right) \geq 0 \quad \Rightarrow \quad |\vec{P}| \leq 1.$$

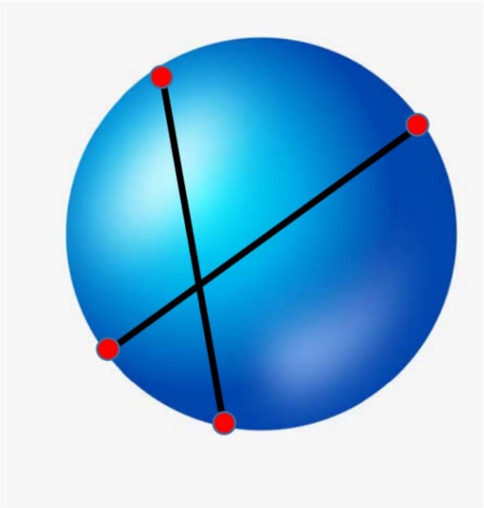
nonnegative

$$\rho(\hat{n}) = \frac{1}{2}(I + \hat{n}\cdot\vec{\sigma}).$$

$$\rho(\vec{P}) = \frac{1}{2}\left(I + \vec{P}\cdot\vec{\sigma}\right) \Rightarrow \mathrm{tr}\rho(\vec{P})(\hat{n}\cdot\sigma) = \hat{n}\cdot\vec{P}.$$

Polarization of the qubit

Density operator of a qubit



Every mixed state of a qubit
=The convex combination of two pure states

(in many ways)

A unique way to express (almost any) mixed state:

A convex combination of two mutually orthogonal pure states

(except the center of the ball)